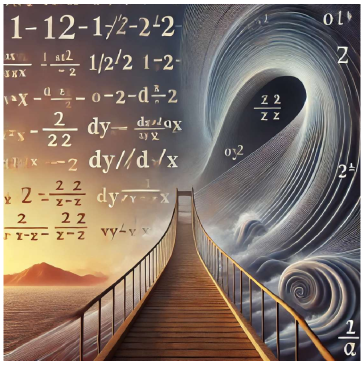
Probability Theory

Textbook: *Introduction to Probability* by Blitzstein and Hwang

Previous Lecture

- Median/Mode
- Moments: Central/Standardized
- Skewness/Kurtosis
- Sample Moments/Law of Large Numbers

§6.4 - Moment Generating Functions (MGFs)



Generating functions are a bridge between **sequences of numbers** and the world of **calculus**. Start with a sequence of numbers, then create a continuous function that encodes the sequence – the generating function. We then have all the tools of calculus at our disposal for manipulating the generating function.

A moment generating function exists that encodes the moments $E(X^n)$ of a distr.

Def (**Moment Generating Function**): If it exists, the MGF of X is $M(t) := E(e^{tX})$ as a function of t.

The MGF exists if $M(t)$ is finite on some open interval $(-a, a)$ containing 0.

The inclusion of t let's us use calculus!

❗ $M(0) = 1$ for any valid MGF M. So, whenever you compute an MGF, plug-in 0 and see if you get 1, as a quick check!

Ex (**Bernoulli MGF**): For $X \sim Bern(p)$, which two values does e^{tX} takes?

The value e^t w/prob p and the value 1 w/prob q, so $M(t) =$??

$M(t) = E(e^{tX}) = pe^t + q$.

Since this is finite for all values of t, the MGF is defined on $\mathbb{R}$. $(M(0) = 1$?)

Ex (**Geometric MGF**): For $X \sim Geom(p)$, the MFG is $M(t) =$??

$M(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} q^k p =$

$= p\sum_{k=0}^{\infty}(qe^t)^k = \frac{p}{1-qe^t}$,

for $qe^t < 1$, i.e., for t in $(-\infty, log\frac{1}{q})$, which is an open interval containing 0. ($M(0) = 1$?)

Ex (**Uniform MGF**): For $U \sim Unif(a,b)$, the MGF is $M(t) = ??$

$M(t) = E(e^{tU}) = \frac{1}{b-a}\int_a^b e^{tu}du = \frac{1}{b-a}[\frac{e^{tu}}{t}]_a^b = \frac{e^{tb}-e^{ta}}{t(b-a)}$ (restrictions?)

for $t \neq 0$, and $M(0) = 1$.

MGFs Features:

- Encodes a rv's moments.
- Determines a rv's distr, (like CDF and PF) via it's moments.
- Makes it easy to find the distr of a sum of indep rvs.

Thm (**Moments via Derivatives of the MGF**): Given the MGF of X, we can obtain the nth moment of X by evaluating the nth derivative of the MGF at 0. So $E(X^n) = M^{(n)}(0)$.

Proof: Note that the Taylor expansion of $M(t)$ about 0 is $M(t) = \sum_{n=0}^{\infty} M^{(n)}(0)\frac{t^n}{n!}$.

On the other hand, we also have $M(t) = E(e^{tX}) = E\left(\sum_{n=0}^{\infty} X^n \frac{t^n}{n!}\right)$. (using taylor exp of e^{tX})

Next, we're allowed to interchange the position of the expectation and the infinite sum above because certain technical conditions are satisfied. So $M(t) = \sum_{n=0}^{\infty} E(X^n)\frac{t^n}{n!}$.

Matching the cofficients of the two expansions, we get $E(X^n) = M^{(n)}(0)$. ■

So, w/the MGF, it is possible to find moments by taking derivatives rather than doing integrals!

Thm (**MGF Determines the Distr**). The MGF of a rv determines its distr. if two rvs have the same MGF, they have the same distr.

If there's even **a tiny interval** $(-a, a)$ containing 0 on which the MGFs are equal, the rvs must have the same distr.

[Proof Requires Analysis]

Thm (**MGF of a Sum of Indep Rvs**): If X and Y are indep, then the MGF of $X + Y$ is the product of the individual MGFs: $M_{X+Y}(t) = M_X(t)M_Y(t)$.

This is true because if X and Y are indep, then $E(e^{t(X+Y)}) = E(e^{tX}e^{tY}) = E(e^{tX})E(e^{tY})$ (this follows from thm 7.3.2). ■

Using this fact, we can get the MGFs of the Binomial and Negative Binomial, which are sums of indep Bernoullis and Geometrics, respectively.

Proposition (**MGF of Location-Scale Transformation**): If X has MGF $M_X(t)$, then the MGF of $bX + a$ is:

$$E(e^{t(bX+a)}) = E(e^{at}e^{btX})$$

$$= e^{at}E(e^{btX}) = e^{at}M_X(bt). \quad \blacksquare$$

Ex (**Moments via Derivatives of MGF**): Let X have the PF: $f(x) := \begin{cases} \frac{2}{11} & \text{if } X = -1, \\ \frac{3}{11} & \text{if } X = 0, \\ \frac{4}{11} & \text{if } X = 1, \\ \frac{2}{11} & \text{if } X = 2, \\ 0 & \text{otherwise.} \end{cases}$

a. Find $M_X(t)$.

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=-1}^{2} e^{tx}f(x) = \tfrac{2}{11}e^{-1t} + \tfrac{3}{11}e^{0t} + \tfrac{4}{11}e^{t} + \tfrac{2}{11}e^{2t}$$

$$= \tfrac{3}{11} + \tfrac{2}{11}e^{-t} + \tfrac{4}{11}e^{t} + \tfrac{2}{11}e^{2t}.$$

b. Using $M_X(t)$, find $E(X)$.

$$M_X'(t) = -\tfrac{2}{11}e^{-t} + \tfrac{4}{11}e^{t} + \tfrac{4}{11}e^{2t}.$$

$$E(X) = M_X'(0) = -\tfrac{2}{11}e^{0} + \tfrac{4}{11}e^{0} + \tfrac{4}{11}e^{2\cdot 0} = \tfrac{6}{11}.$$

c. Let $Y = 7X - 5$. Use $M_X(t)$ to find $M_Y(t)$.

$$M_Y(t) = E(e^{tY}) = E(e^{7Xt-5t}) = E(e^{7Xt}e^{-5t}) = e^{-5t}M_X(7t)$$

$$= e^{-5t}\left(\tfrac{3}{11} + \tfrac{2}{11}e^{-7t} + \tfrac{4}{11}e^{7t} + \tfrac{2}{11}e^{14t}\right)$$

$$= \tfrac{3}{11}e^{-5t} + \tfrac{2}{11}e^{-12t} + \tfrac{4}{11}e^{2t} + \tfrac{2}{11}e^{9t}.$$

Ex (Uniform MGF):

a. Find the MGF ($M_U(t)$) of $U \sim Unif(0,1)$.

$$M_U(t) = E(e^{tU})$$

$$= \frac{1}{1-0}\int_0^1 e^{tu}du = \frac{e^t-1}{t} \text{ for } t \neq 0.$$

b. Using M_X from the previous example, and assuming U and X are indep find $M_{U+X}(t)$.

$$M_{U+X}(t) = M_U(t)M_X(t) = \frac{e^t-1}{t}\left(\frac{3}{11} + \frac{2}{11}e^{-t} + \frac{4}{11}e^t + \frac{2}{11}e^{2t}\right).$$

c. Use $M_U(t)$ to find the mean and variance of U.

$$M_U'(t) = \frac{te^t-(e^t-1)}{t^2} = \frac{te^t-e^t+1}{t^2}.$$

$$M_U''(t) = \frac{e^t}{t} - \frac{2(t-1)e^t+2}{t^3}.$$

$$E(X) = M_U'(0) = \frac{te^t-e^t+1}{t^2}\Big|_{t=0}$$

$$\overset{L'H}{=} \frac{te^t}{2t}\Big|_{t=0}$$

$$\overset{L'H}{=} \frac{e^t+te^t}{2}\Big|_{t=0}$$

$$= \frac{2-1}{2} = \frac{1}{2}.$$

$$E(X^2) = M_U''(t) = \left(\frac{e^t}{t} - \frac{2(te^t-e^t+1)}{t^3}\right)\Big|_{t=0} = \left(\frac{t^2e^t-2te^t+2e^t-2}{t^3}\right)\Big|_{t=0}$$

$$\overset{L'H}{=} \left(\frac{e^t}{3}\right)\Big|_{t=0} = \frac{1}{3}.$$

So, $Var(X) = E(X^2) - E(X)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$

Activity 13

Harvard Videos:

youtube.com/watch?v=N8O6zd6vTZ8&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=18

youtube.com/watch?v=tVDdx6xUOcs&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=19

youtube.com/watch?v=xiVWNkQUqKk&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=20

What did we learn?

- Moment Generating Functions (MGFs)
- Moments via Derivatives of MGFs
- MGF of Location-Scale Transformation