

Applied Linear Algebra

Textbook: *Applied Linear Algebra* by Oliver and Shakiban

8.5 Eigenvalues of Symmetric Matrices

Good news, most matrices found in application are symmetric, AND have e-vecs which form an orthogonal basis for the underlying space (complete). Let's see what else this gives us:

Theorem: Let $\mathbf{A} = \mathbf{A}^T$ be a real symmetric $n \times n$ matrix. Then,

- All the e-vals of \mathbf{A} are *real*
- E-vecs corresponding to *distinct* e-vals are *orthogonal*.
- There's an orthonormal basis of \mathbb{R}^n consisting of n e-vecs of \mathbf{A} .

In particular, all real symmetric matrices are *complete* and real *diagonalizable*.

Proof quite long, in book.

Concretely: Find an orthonormal basis of \mathbb{R}^n given $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, that has $\lambda_1 = 4$, $\lambda_2 = 2$ and $\vec{v}_1 = (1, 1)$, $\vec{v}_2 = (-1, 1)$.

Note \mathbf{A} is symmetric. So, it is easily checked that $\vec{v}_1 \cdot \vec{v}_2 = 0$.

To get the orthonormal basis, we need only normalize the \vec{v}_i : $\vec{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\vec{u}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Theorem: A symmetric $\mathbf{K} = \mathbf{K}^T$ is positive definite **iff** all of its e-vals are strictly positive.

Proof: (\Rightarrow) First, if $\mathbf{K} > 0$, then, by definition, $\vec{x}^T \mathbf{K} \vec{x} > 0$ for all nonzero $\vec{x} \in \mathbb{R}^n$.

In particular, if $\vec{x} = \vec{v} \neq \vec{0}$ is an e-vec with (necessarily real) e-val λ , then

$$0 < \vec{v}^T \mathbf{K} \vec{v} = \vec{v}^T (\lambda \vec{v}) = \lambda \vec{v}^T \vec{v} = \lambda |\vec{v}|^2,$$

which immediately proves $\lambda > 0$.

Conversely (\Leftarrow), suppose \mathbf{K} has all positive e-vals. Let $\vec{u}_1, \dots, \vec{u}_n$ be the orthonormal e-vec basis guaranteed by the previous thm, with $\mathbf{K} \vec{u}_j = \lambda_j \vec{u}_j$ where $\lambda_j > 0$.

Writing $\vec{x} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n$, we obtain $\mathbf{K} \vec{x} = c_1 \lambda_1 \vec{u}_1 + \dots + c_n \lambda_n \vec{u}_n$.

Therefore, using orthonormality of the e-vecs:

$$\vec{x}^T \mathbf{K} \vec{x} = (c_1 \vec{u}_1^T + \dots + c_n \vec{u}_n^T)(c_1 \lambda_1 \vec{u}_1 + \dots + c_n \lambda_n \vec{u}_n) = \lambda_1 c_1^2 + \dots + \lambda_n c_n^2.$$

Note this last expression is greater than zero whenever $\vec{x} \neq \vec{0}$,

since only $\vec{x} = \vec{0}$ has coordinates $c_1 = \dots = c_n = 0$. This establishes $\mathbf{K} > \mathbf{0}$. ■

Proposition: Let $\mathbf{A} = \mathbf{A}^T$ be $n \times n$, symmetric. Let $\vec{v}_1, \dots, \vec{v}_n$ be an orthogonal e-vec basis such that $\vec{v}_1, \dots, \vec{v}_r$ correspond to nonzero e-vals, while $\vec{v}_{r+1}, \dots, \vec{v}_n$ are null e-vecs corresponding to the zero e-val (if any). Then $r = \text{rank } \mathbf{A}$; the non-null e-vecs $\vec{v}_1, \dots, \vec{v}_r$ form an orthogonal basis for $\text{img } \mathbf{A} = \text{coimg } \mathbf{A}$, while the null e-vecs $\vec{v}_{r+1}, \dots, \vec{v}_n$ form an orthogonal basis for $\ker \mathbf{A} = \text{coker } \mathbf{A}$.

Proof: The zero e-space coincides with kernel: $V_0 = \ker(\mathbf{A} - \mathbf{0I}) = \ker \mathbf{A}$.

Thus, the linearly independent null e-vecs form a basis for $\ker \mathbf{A}$, which has dimension $n - r$ where $r = \text{rank } \mathbf{A}$.

Moreover, the remaining r non-null e-vecs are (by prev. thm) orthogonal to the null e-vecs. Therefore, they must

form a basis for the kernel's orthogonal complement, namely $\text{coimg } \mathbf{A} = \text{img } \mathbf{A}$. (since $\mathbf{A} = \mathbf{A}^T$) ■

Example: Determine whether $\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{bmatrix}$ is positive definite by computing its e-vals.

Validate your conclusions by using the methods learned previously in the course.

$$\begin{aligned} \text{Observe the symmetry. Then: } & \begin{vmatrix} 4 - \lambda & -1 & -2 \\ -1 & 4 - \lambda & -1 \\ -2 & -1 & 4 - \lambda \end{vmatrix} = \begin{vmatrix} 4 - \lambda & -1 & -2 \\ -1 & 4 - \lambda & -1 \\ 0 & 2\lambda - 9 & 6 - \lambda \end{vmatrix} \\ & = (4 - \lambda)((4 - \lambda)(6 - \lambda) - (9 - 2\lambda)) - (-1)(-(6 - \lambda) - (-2)(2\lambda - 9)) = -\lambda^3 + 12\lambda^2 - 42\lambda + 36. \end{aligned}$$

Note the factors of (the constant term) 36 are $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 13, \pm 36\}$.

Try these as possible solutions until eventually, finding $\lambda = 6$.

Polynomial division by $(\lambda - 6)$.

$$\begin{aligned} -\lambda^3 + 12\lambda^2 - 42\lambda + 36 &= (\lambda - 6)(-\lambda^2) + (6\lambda^2 - 42\lambda + 36) = (\lambda - 6)(-\lambda^2) + (\lambda - 6)(6\lambda) + (-6\lambda + 36) \\ &= (\lambda - 6)(-\lambda^2) + (\lambda - 6)(6\lambda) - 6(\lambda - 6). \end{aligned}$$

$$\text{So, } -\lambda^3 + 12\lambda^2 - 42\lambda + 36 = (\lambda - 6)(-\lambda^2 + 6\lambda - 6),$$

$$\text{and } \lambda \in \{6, 3 \pm \sqrt{3}\} \approx \{6, 1.27, 4.73\}. \quad (\text{quadratic formula})$$

Pos e-vals, so pos def.

Verify w/ "other method in course"?

Recall, a symmetric matrix is positive definite **iff** it is regular and has all positive pivots.

$$\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & \frac{15}{4} & -\frac{3}{2} \\ 0 & -9 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & \frac{15}{4} & -\frac{3}{2} \\ 0 & 0 & \frac{16}{5} \end{bmatrix}, \text{ so pos def.}$$

The Spectral Theorem

The previous section told us that a real symmetric matrix produces an e-basis for the space, and is therefore diagonalizable. Furthermore, since we can choose those e-vecs such that they are of unit length, we have the following:

Theorem: Let \mathbf{A} be real, symmetric. Then there exists an orthogonal \mathbf{Q} such that $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where $\mathbf{\Lambda}$ is real/diagonal. The e-vals of \mathbf{A} appear on the diagonal of $\mathbf{\Lambda}$, while the columns of \mathbf{Q} are the corresponding orthonormal e-vecs.

! $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ is not the same as $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^{-1}$ (the latter produced from row reduction method).

The spectral factorization provides us an alternative means of diagonalizing the associated quadratic form $q(\vec{x}) = \vec{x}^T \mathbf{A} \vec{x}$.

That is, a means for completing the square.

$$\text{Observe (for real symmetric } \mathbf{A}\text{): } q(\vec{x}) = \vec{x}^T \mathbf{A} \vec{x} = \vec{x}^T \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} \vec{x} = \vec{y}^T \mathbf{\Lambda} \vec{y} = \sum_{i=1}^n \lambda_i y_i^2,$$

where $\vec{y} = \mathbf{Q}^T \vec{x} = \mathbf{Q}^{-1} \vec{x}$ are the coords of \vec{x} with respect to the orthonormal e-vec basis of \mathbf{A} .

In particular, $q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$ and so \mathbf{A} is positive definite **iff** each e-val is strictly positive,

reconfirming our previous thm.

Example: Construct a symmetric \mathbf{A} with the following e-vecs and e-vals, or explain why none exists:

$$\lambda_1 = -2, \quad \vec{v}_1 = (1, -1) \text{ and } \lambda_2 = 1, \quad \vec{v}_2 = (1, 1).$$

It's diagonalizable (real, symmetric, so two e-vecs), and forming $\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$, we have:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

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