

# Applied Linear Algebra

Textbook: *Applied Linear Algebra* by Oliver and Shakiban

## 7.2 Linear Transformations

Consider linear function  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that maps  $n$ D Euclidean space to itself.  $L$  maps a  $\vec{x} \in \mathbb{R}^n$  to  $L[\vec{x}] = \mathbf{A}^{n \times n} \vec{x}$ .

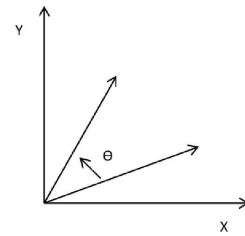
When thinking of it geometrically like this, we refer to  $L$  as a linear transformations (LT).

In  $\mathbb{R}^2$ , they have the form: 
$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix},$$

where  $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

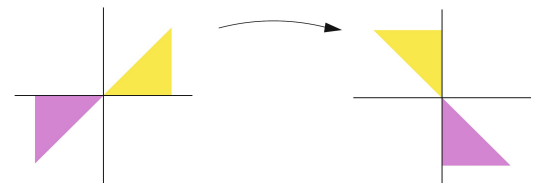
### Types of LTs in $\mathbb{R}^2$

**Rotation Matrices:**  $\mathbf{R}_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , rotating vectors through angle  $\theta$ .



Rotations **without reflections** occur with proper orthogonal matrices.

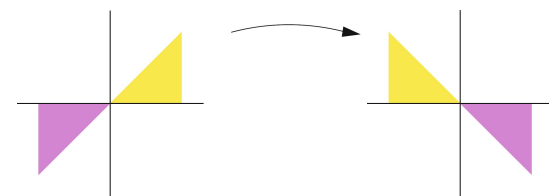
That is, matrices  $\mathbf{Q}$  that satisfy:  $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$  and  $\det \mathbf{Q} = +1$ .



Rotations without reflections

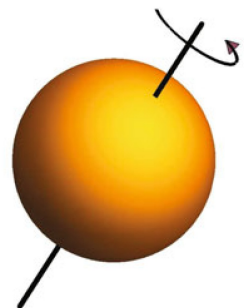
Alternatively, rotations **with reflections** occur with improper orthogonal matrices:

$\mathbf{Q}$  Such that  $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$  and  $|\mathbf{Q}| = -1$ .



Rotations with reflections

A proper 3D orthogonal matrix  $\mathbf{Q} = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]$  with  $\vec{u}_i = \mathbf{Q} \hat{e}_i$  corresponds to rotation in which  $\hat{e}_i$  are rotated to new positions given by orthonormal basis  $\vec{u}_i$ . Every  $3 \times 3$  orthogonal matrix corresponds to rotation around a line through the origin in  $\mathbb{R}^3$  (axis of rotation).



Since the product of two (proper) orthogonal matrices is also (proper) orthogonal, the composition of two rotations is also a

rotation.

Unlike 2D case, order in which rotations are performed in 3D is important.

Multiplication of  $n \times n$  orthogonal matrices is not commutative when  $n \geq 3$ .

**Example:** Rotating first around  $y$ -axis, then around  $x$ -axis, and then  $x$ -axis.

Not the same as around  $x$ -axis, then  $y$ -axis, then  $x$ -axis.



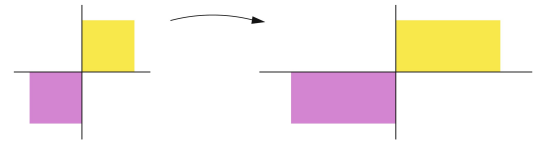
$\frac{\pi}{2}$  around  $y$ , then  $x$ , then  $x$ .       $\frac{\pi}{2}$  around  $x$ , then  $y$ , then  $x$ .

see animation in class

## LTs from elementary matrices:

♦ Multiplying a row by a scalar  $\Rightarrow$  stretching and/or reflection transformation.

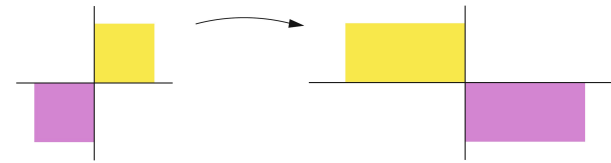
**Example:**  $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}.$



stretching

**Example:**  $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$

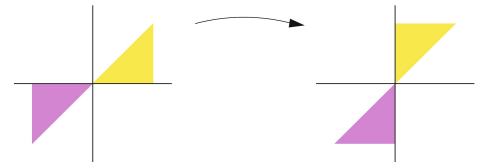
corresponds to reflection through  $y$ -axis followed by previous stretching.



reflection through  $y$ -axis then stretching

♦ Row interchange  $\Rightarrow$  reflection through diagonal  $y = x$ .

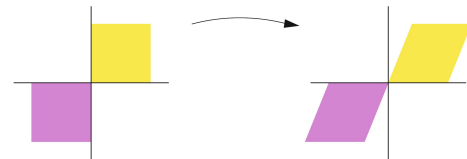
**Example:**  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$



reflection through  $y = x$

♦ Adding a multiple of one row to another  $\Rightarrow$  Shear along some axis.

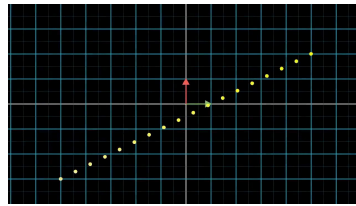
**Example:**  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ y \end{bmatrix}.$



Shear

Noninvertible LTs  $\Rightarrow$  projection from the plane onto the origin, or onto a line through the origin.

**Example:**  $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 0 \end{bmatrix}.$



See animation in class

## Generalizing

In higher dimensions, elementary matrices can produce LTs of types:

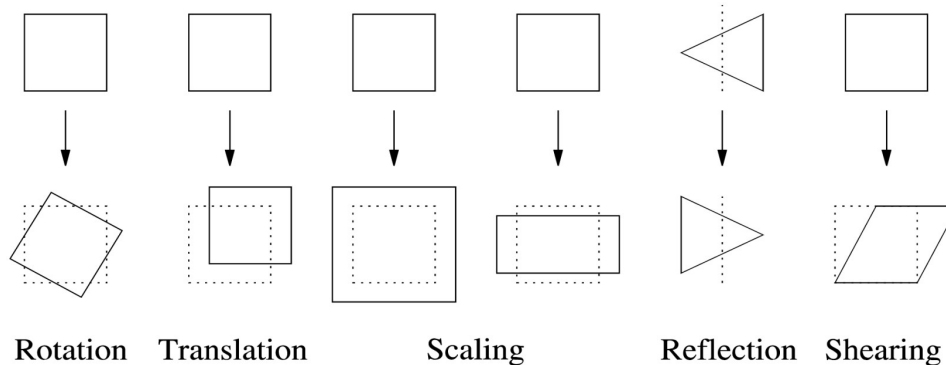
- ◆ **stretch** in a single coordinate direction.
- ◆ **reflect** through a coordinate (hyper)plane.
- ◆ **reflect** through a diagonal (hyper)plane.
- ◆ **shear** along a coordinate axis.

And *all* invertible LTs can be constructed from a sequence of these types.

The last type of LT is noninvertible, and so is not the result of an elementary matrix:

- ◆ orthogonally **project** onto a lower dimensional subspace.

And *ALL* transformations can be built up from the previous five types.



## Change of Basis

Up until now, vector notation  $\langle x_1, x_2, x_3 \rangle$  was meant to indicate:  $x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$ .

Indeed, when you have seen matrix multiplication  $\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \vec{x}$ , we have intended it to mean:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} (x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3) \quad (\text{Std unit basis } \hat{e}_1, \hat{e}_2, \hat{e}_3 \text{ in domain})$$

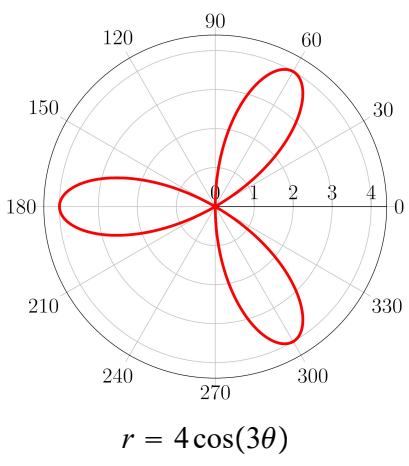
$$= x_1 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \hat{e}_1 + x_2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \hat{e}_2 + x_3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \hat{e}_3$$

$$= x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= (x_1 + 2x_2 + 3x_3)\hat{e}'_1 + (4x_1 + 5x_2 + 6x_3)\hat{e}'_2. \quad (\text{Std unit basis } \hat{e}'_1, \hat{e}'_2 \text{ in codomain})$$

However, observe that this relies on the choice of standard unit basis vectors.

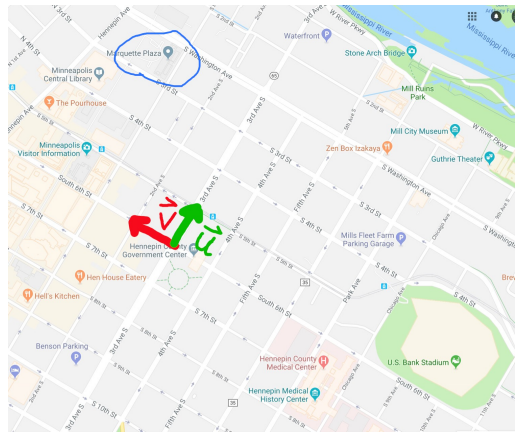
In some applications, one can gain additional insight, or create computational efficiency by adopting a different basis.



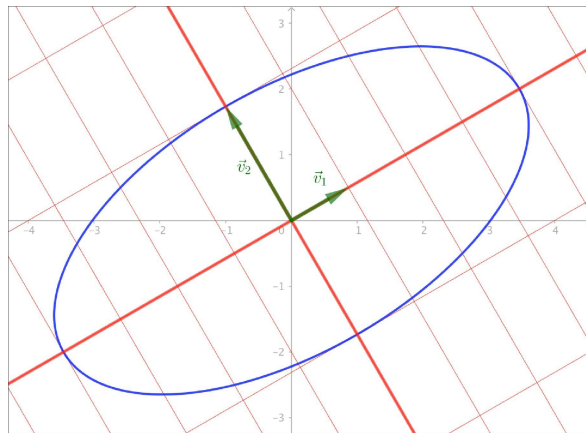
Polar coordinates make this shape simple to express.

In cartesian coordinates, we have:  $x^4 + y^4 + 2x^2y^2 - 4x^3 + 12xy^2 = 0$ . (!?!)

Similarly, in linear algebra, coordinates can make all the difference.



$\vec{u}, \vec{v}$ -coordinates better than  $\hat{e}_1, \hat{e}_2$



Geometry may suggest convenient coordinates

**Definition:** If  $\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ , where  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ , then we say  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$  are the  $\mathcal{B}$ -coordinates of  $\vec{x}$ .

$\vec{x}$  in *standard coordinates* is denoted simply as  $\vec{x}$ .

$\vec{x}$  in  $\mathcal{B}$ -coordinates is denoted  $[\vec{x}]_{\mathcal{B}}$ . ...

**Theorem:** Let  $L : V \rightarrow W$  be a linear function. Suppose  $V$  has basis  $\mathcal{B}_v = \{\vec{v}_1, \dots, \vec{v}_n\}$  and  $W$  has basis  $\mathcal{B}_w = \{\vec{w}_1, \dots, \vec{w}_m\}$ . We can write:  $\vec{v} = x_1\vec{v}_1 + \dots + x_n\vec{v}_n \in V$ ,  $\vec{w} = y_1\vec{w}_1 + \dots + y_m\vec{w}_m \in W$ , where  $\vec{x} = (x_1, \dots, x_n)^T$  are the coordinates of  $\vec{v}$  **relative to the basis of  $V$**  and  $\vec{y} = (y_1, \dots, y_m)^T$  are those of  $\vec{w}$  **relative to the basis of  $W$** . Then, in those coordinates, there exists  $\mathbf{S}^{m \times n}$  such that the linear function  $\vec{w} = L[\vec{v}]$  is given by multiplication by  $\mathbf{S}$ , so  $\vec{y} = \mathbf{S}\vec{x}$ .

**Proof:** We mimic proof of Theorem 7.5 (previous section), in which we discovered the matrix representative by

$$L[\vec{x}] = \mathbf{A}\vec{x} = [L[\vec{e}_1] \dots L[\vec{e}_n]]\vec{x} = x_1L[\vec{e}_1] + \dots + x_nL[\vec{e}_n].$$

This time, we'll be replacing the  $\vec{e}_1$  with more general basis vectors.

In other words, we apply  $L$  to the basis vectors of  $V$  and express the result as a linear combination of the basis vectors in  $W$ . By linearity, we have:

$$\vec{w} = L[\vec{v}] = L[x_1\vec{v}_1 + \dots + x_n\vec{v}_n] \quad (\text{def. of } \vec{v})$$

$$= x_1L[\vec{v}_1] + \dots + x_nL[\vec{v}_n] \quad (\text{linearity})$$

$$= x_1 \sum_{i=1}^m s_{i1}\vec{w}_i + \dots + x_n \sum_{i=1}^m s_{in}\vec{w}_i \quad (L[\vec{v}_i] \text{ live in } W, \text{ and so linear combo of } \vec{w}_1, \dots, \vec{w}_m)$$

$$= \sum_{j=1}^n x_j \sum_{i=1}^m s_{ij}\vec{w}_i = \sum_{j=1}^n \sum_{i=1}^m s_{ij}x_j\vec{w}_i = \sum_{i=1}^m \left( \sum_{j=1}^n s_{ij}x_j \right) \vec{w}_i, \quad (\text{arithmetic})$$

and so we find  $y_i = \sum_{j=1}^n s_{ij}x_j$ , and  $\vec{y} = \mathbf{S}\vec{x}$ , where  $\mathbf{S} = [\vec{s}_1 \dots \vec{s}_n]$  and  $\vec{s}_j = (s_{1j}, \dots, s_{mj})$ , as claimed. ■



**Conversion:  $\mathcal{B}$ -coords  $\rightarrow$  Std** (easy, always possible)

**Example:** Given  $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$ , basis of  $\mathbb{R}^2$ , find std coords for:

◆  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 7 \end{bmatrix} \quad \dots$

$$\vec{x} = -3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -24 \end{bmatrix}.$$

◆  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \quad \dots$

$$\vec{x} = 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}.$$

In general, given  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$  :  $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$

$$= [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{S}[\vec{x}]_{\mathcal{B}}.$$

So  $\vec{x} = \mathbf{S}[\vec{x}]_{\mathcal{B}}$ , where  $\mathbf{S}$ 's columns are the basis vectors.



## Conversion: Std $\rightarrow$ $\mathcal{B}$ -coordinates (hard, not always possible)

**Example:** Again with  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$ , find  $\mathcal{B}$ -coords for  $\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ .

We know  $\vec{x} = \mathbf{S}[\vec{x}]_{\mathcal{B}}$ , i.e.  $\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ . ...

Method 1:  $\left. \begin{array}{l} c_1 + 2c_2 = 7 \\ c_1 - 3c_2 = 2 \end{array} \right\}$  Solve.

Method 2:  $\vec{x} = \mathbf{S}[\vec{x}]_{\mathcal{B}} \Rightarrow [\vec{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\vec{x}$  ...

$$= -\frac{1}{5} \begin{bmatrix} -3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

**Example:**  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  form basis for  $V = \text{span}(\vec{v}_1, \vec{v}_2) \subset \mathbb{R}^3$ .

Find  $[\vec{x}]_{\mathcal{B}}$  if  $\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 10 \end{bmatrix}$  ...

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 10 \end{bmatrix} \Rightarrow \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & 10 \end{array} \right] \dots$$

$$\rightarrow \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 10 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 12 \end{array} \right] \quad (!?!)$$

Resulting system has no solution. ( $\vec{x} \notin V$ ) What does this mean?

**How can  $\mathcal{B}$ -coordinates make my life easier?**

Consider  $T(\vec{x}) = \text{"projection of } \mathbb{R}^2 \text{ onto line } \ell = \{(x,y) : y = \frac{3}{4}x\}$ ".

Turns out,  $T(\vec{x}) = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix} \vec{x}$ . But why?

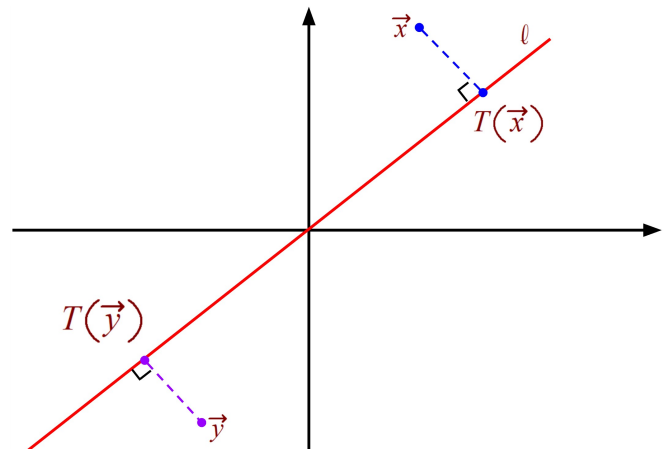
How would we find this? Let's find out.

Choose  $\vec{v}_1$  to span  $\ell$ , and  $\vec{v}_2 \neq \vec{0}$  such that  $\vec{v}_2 \perp \vec{v}_1$ .

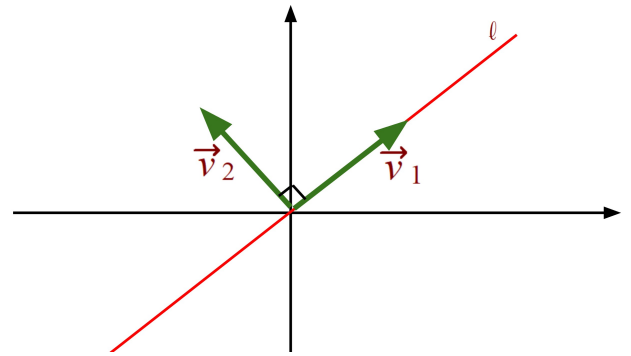
**Example:**  $\vec{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ .

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$  forms a basis of  $\mathbb{R}^2$ .

Any  $\vec{x} \in \mathbb{R}^2$  can be written uniquely as combination of  $\vec{v}_1, \vec{v}_2$ .



$T(\vec{x}) = \text{projection of } \vec{x} \text{ onto line } \ell = \{(x,y) : y = \frac{3}{4}x\}$



$\vec{v}_1 \parallel \ell$  and  $\vec{v}_2 \perp \vec{v}_1$

Notice that with this choice of vectors,  $T(\vec{v}_1) = \vec{v}_1$ , and  $T(\vec{v}_2) = \vec{0}$ .

So for any  $\vec{x} \in \mathbb{R}^2$ ,  $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = c_1\vec{v}_1 + 0\vec{v}_2 = c_1\vec{v}_1$ .

Therefore, in  $\mathcal{B}$ -coords, we send  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  to  $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ .

So in  $\mathcal{B}$ -coords,  $T$  is represented by  $\mathbf{B}$  such that:

$$\mathbf{B}\vec{c} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}.$$

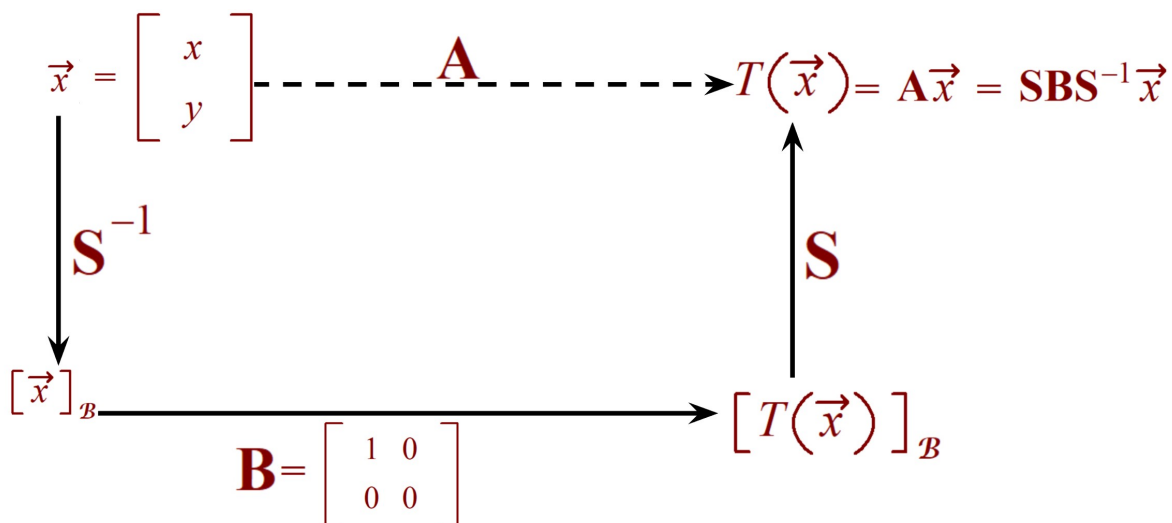
So,  $\mathbf{B}$ , the  $\mathcal{B}$ -matrix of  $T$ , tells us how to perform  $T$  in  $\mathcal{B}$ -coords.

So we found the matrix! But not in the standard basis.

**Q:** How do we recover the "standard"  $\mathbf{A} = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$  from the  $\mathcal{B}$ -matrix  $\mathbf{B}$ ?

Observe that:  $\ell = \text{span} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ ,  $\mathbf{B} = \left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ ,  $\mathbf{S} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ ,  $\mathbf{S}^{-1} = \frac{1}{25} \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ .

Using a "commutative diagram," we can construct the projection onto  $\ell$  in pieces.  
(commutative diagrams can be used to represent function composition)



We say a diagram **commutes** if all "paths" between some start/end points give the same overall function.

So  $\mathbf{A} = \mathbf{SBS}^{-1}$ .

For our line  $\ell$ ,  $T(\vec{x}) = \mathbf{A}\vec{x} = \mathbf{SBS}^{-1}\vec{x} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{25} & \frac{3}{25} \\ -\frac{3}{25} & \frac{4}{25} \end{bmatrix} \vec{x} = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix} \vec{x}$ . (we found it!)

Reflection *across*  $\ell$  would be  $\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{4}{25} & \frac{3}{25} \\ -\frac{3}{25} & \frac{4}{25} \end{bmatrix}$  ( $\vec{v}_2$  sent to  $-\vec{v}_2$ , not  $\vec{0}$ )

**Definition** (similar matrices):  $\mathbf{A}$  is similar to  $\mathbf{B}$ , written  $\mathbf{A} \sim \mathbf{B}$ , if they represent same linear transformation, but possibly in a different basis.

Algebraically:  $\mathbf{A} = \mathbf{SBS}^{-1} \Rightarrow \mathbf{AS} = \mathbf{SB} \Rightarrow \mathbf{S}^{-1}\mathbf{AS} = \mathbf{B}$ .

**Example:** From above:  $\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .

**Recall:** Given linear transformation  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,

matrix representation of  $L$  is:  $\mathbf{A} := \begin{bmatrix} | & | & & | \\ L(\vec{e}_1) & L(\vec{e}_2) & \dots & L(\vec{e}_m) \\ | & | & & | \end{bmatrix}$ .

**Example:** Find matrix form  $\mathbf{B}$  of  $L(x,y) = \begin{bmatrix} x - 4y \\ -2x + 3y \end{bmatrix}$  with respect to basis  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

$$\mathbf{S} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{A} = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}.$$

$$\text{Therefore: } \mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}.$$

**Verifying:** Let's characterize  $\vec{e}_1, \vec{e}_2$  in  $\mathcal{B}$ -coords, apply  $\mathbf{B}$ , and see if  $\mathbf{B}$  send these vectors to the same location as  $\mathbf{A}$  does.

$$\text{Notice } \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \mathbf{S}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = [\vec{e}_1]_{\mathcal{B}} \text{ and } \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = [\vec{e}_2]_{\mathcal{B}}.$$

$$\text{Applying } \mathbf{B}, \text{ we get: } \mathbf{B}[\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix},$$

$$\text{and } \mathbf{B}[\vec{e}_2]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix}.$$

$$\text{In std coords: } \mathbf{B}[\vec{e}_1]_{\mathcal{B}} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{v}_1$$

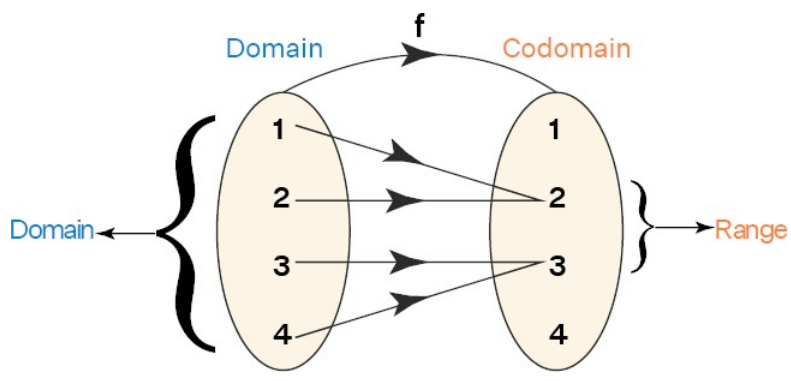
$$\mathbf{B}[\vec{e}_2]_{\mathcal{B}} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{7}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \vec{v}_2 \quad \checkmark$$

Videos explaining linear algebra with visually spectacular and intuitive explanations, see:

[Youtube.com/watch?v=P2LTAUO1TdA&list=PLZHQObOWTQDPD3MizzM2xVFItgF8hE\\_ab](https://www.youtube.com/watch?v=P2LTAUO1TdA&list=PLZHQObOWTQDPD3MizzM2xVFItgF8hE_ab)

# Other Potentially Useful Materials

(used in previous classes I've taught)



A function  $f : X \rightarrow Y$  contains 3 pieces of information:

- ◆ Domain:  $X$
- ◆ Codomain:  $Y$
- ◆ A rule sending elements from domain to codomain:  $f$

**Example:**  $f(x) = x^2$ . New notation:  $f : \mathbb{R} \rightarrow \mathbb{R}$  where  $x \mapsto x^2$  ( $x$  maps to  $x^2$ ).

**Example:**  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  where 
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} \sin x \\ y + z \end{bmatrix}.$$

## Linear Transformation, a special type of function

**Linear Transformation:**  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a function which takes in values from  $\mathbb{R}^m$  (vectors) and outputs values in  $\mathbb{R}^n$  (vectors) and also satisfies  $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$ , for all  $a, b \in \mathbb{R}$  and  $\vec{u}, \vec{v} \in \mathbb{R}^m$ .

**Recall:** A linear combination of vectors  $\vec{u}, \vec{v}$  is a vector of the form  $a\vec{u} + b\vec{v}$ .

Linear transformations are functions which "preserve" linear combinations.

**Example:** Is  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  where 
$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y - x \\ y \end{bmatrix}$$
 a linear transformation? ...

$$\begin{aligned} & T\left(a \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{bmatrix}\right) \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} ax_1 + bx_2 \\ (ay_1 + by_2) - (ax_1 + bx_2) \\ ay_1 + by_2 \end{bmatrix} \\
&= a \begin{bmatrix} x_1 \\ y_1 - x_1 \\ y_1 \end{bmatrix} + b \begin{bmatrix} x_2 \\ y_2 - x_2 \\ y_2 \end{bmatrix} \\
&= aT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + bT\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right).
\end{aligned}$$

**Example:** Is  $f(x) = 3x$  a linear transformation? ...

$$f(ax + by) = 3(ax + by) = 3ax + 3by = a3x + b3y = af(x) + bf(y).$$

**Example:** Is  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x^2 \\ 0 \end{bmatrix}$  a linear transformation? ...

$$\text{Observe: } T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{However: } T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad !?!$$

**Linear Transformations Theorem:**  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is **linear** if (and only if):

- ♦  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ , for all  $\vec{v}, \vec{w} \in \mathbb{R}^m$ , and
- ♦  $T(k\vec{v}) = kT(\vec{v})$ , for all  $\vec{v} \in \mathbb{R}^m$  and all scalars  $k$ .

**Example:** Let  $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$  and  $\mathbf{A} \in \mathbb{R}^{n \times m}$ .

$$\text{Observe: } \vec{x} = x_1 \vec{e}_1 + \dots + x_m \vec{e}_m.$$

$$\text{So, } \mathbf{A}\vec{x} = \mathbf{A}(x_1 \vec{e}_1 + \dots + x_m \vec{e}_m) = \mathbf{A}(x_1 \vec{e}_1) + \dots + \mathbf{A}(x_m \vec{e}_m) = x_1 \mathbf{A}(\vec{e}_1) + \dots + x_m \mathbf{A}(\vec{e}_m). \quad (\text{Chap. 1})$$

All information about linear transformations is encoded in where the transformation sends the basis vectors  $\vec{e}_i$ .

In  $\mathbb{R}^2$ , using standard basis:

$$\mathbf{A}\vec{e}_1 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \text{ and } \mathbf{A}\vec{e}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So, a linear transformation sends the 1st basis vector to the 1st column vector, and the 2nd basis vector to the 2nd column.

**Matrix Columns of a Linear Transformation Theorem:** Given linear transformation  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ .

$$\text{Matrix of } T \text{ is: } \mathbf{A} = \begin{bmatrix} | & | & \dots & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & \dots & | \end{bmatrix}.$$

**Linear Transformation Matrix Correspondence:**  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a linear transformation  $\Leftrightarrow T(\vec{x}) = \mathbf{A}\vec{x}$ , for some  $\mathbf{A}^{n \times m}$ .

**Proof for  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ :** First we show  $\Rightarrow$ .

$$\text{Suppose } T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

$$\text{Note if } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \mathbf{A} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \mathbf{A} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Now we show  $\Leftarrow$ .

Recall  $\mathbf{A}\vec{x} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$  is a linear transformation (as seen earlier),

and its effect on  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is the **same** as  $T$ .

So  $\mathbf{A}\vec{x}$  is a linear transformation. ■

**LT, Zero Identity Thm:** If  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , then  $T(\vec{0}) = \vec{0}$ .

**Proof:** Let  $\vec{0} \in \mathbb{R}^m$  and  $T$  be a linear transformation.

Then,  $T(\vec{0}) = \dots$

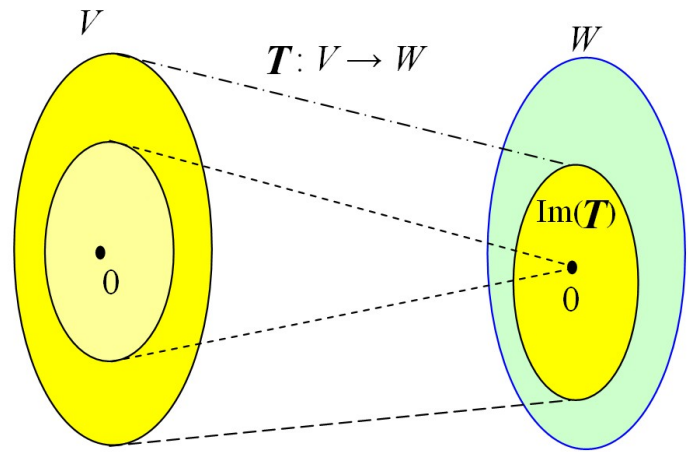
$$= T(0 \cdot \vec{0})$$

$$= 0 \cdot T(\vec{0}) = \vec{0} \in \mathbb{R}^n.$$

Alternatively:  $T(\vec{0})$

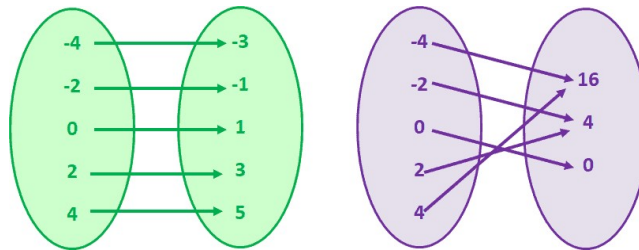
$$= T(\vec{x} - \vec{x})$$

$$= T(\vec{x}) - T(\vec{x}) = \vec{0} \in \mathbb{R}^n.$$



**Corollary:** The Contrapositive.

## Invertibility



**Invertible Transformation:**  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  where there exists a map  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that  $T(S(\vec{x})) = \vec{x}$  and  $S(T(\vec{y})) = \vec{y}$ .

We usually denote the "inverse"  $S$  as  $T^{-1}$ .

**Example:** Show that  $T : \mathbb{R} \rightarrow \mathbb{R}$  where  $x \mapsto 3x$  and  $S : \mathbb{R} \rightarrow \mathbb{R}$  where  $x \mapsto \frac{1}{3}x$  are inverses. ...

$$T(S(x))$$

$$= T\left(\frac{1}{3}x\right)$$

$$= 3 \cdot \left(\frac{1}{3}x\right) = x \quad \text{and} \quad \dots$$

$$S(T(x)) = S(3x)$$

$$= \frac{1}{3}(3x) = x. \quad \blacksquare$$

**Example:** Does  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ x \end{bmatrix}$  have an inverse? ...

Observe that:  $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

If  $S$  exists, then  $S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , and  $S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Whoops.

## Practice

Which are linear transformations? Briefly justify.

a)  $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 3y \\ x - y \\ y \end{bmatrix}$     b)  $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 3y \\ x - y \\ y + 1 \end{bmatrix}$     c)  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 3xy \\ 5yz \\ 7xz \end{bmatrix}$ . ...

For each which is a linear transform, find the matrix representing it.

For a):  $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ , so  $\mathbf{A} = \begin{bmatrix} | & | \\ T\vec{e}_1 & T\vec{e}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ .