

Linearity

Up until now, we've exploited linearity mostly in \mathbb{R}^n , but now we'll generalize, showing its influence in other vector spaces. Indeed, this concept contributes to differential equations, integral equations, stochastic systems, and even quantum mechanics.

7.1 Linear Functions

Definition: Let V and W be real vector spaces. A function $L : V \rightarrow W$ is called *linear* if it obeys two basic rules:

$$L[\vec{v} + \vec{w}] = L[\vec{v}] + L[\vec{w}], \quad L[c\vec{v}] = cL[\vec{v}], \quad \text{for all } \vec{v}, \vec{w} \in V \text{ and all scalar } c.$$

We will call V the domain and W the codomain for L .

Or more succinctly, a linear function respects linear combinations:

$$L[c\vec{v} + d\vec{w}] = cL[\vec{v}] + dL[\vec{w}], \quad \text{for all } \vec{v}, \vec{w} \in V, \quad c, d \in \mathbb{R}. \quad (*)$$

Algebraic Linear Functions

Zero function: $O[\vec{v}] = \vec{0}$.

Identity Function: $I[\vec{v}] = \vec{v}$.

Scalar Multiplication: $M_a[\vec{v}] = a\vec{v}$ for $a \in \mathbb{R}$.

NOT Linear Functions: Even though $y = L[x] = ax + b$ is a straight line, it is not a linear fn when $b \neq 0$.

$$\text{Indeed, } L[x_1 + x_2] = a(x_1 + x_2) + b$$

$$\neq (ax_1 + b) + (ax_2 + b) = L[x_1] + L[x_2].$$

Also: $L[0] = b \neq 0$.

Instead, these are called *affine functions*.

Matrices: Matrices are easily seen as linear functions.

Observe that $\mathbf{A}[c\vec{v} + d\vec{w}] = c\mathbf{A}\vec{v} + d\mathbf{A}\vec{w}$.

In fact:

Thm: Every linear function $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (between finite dim spaces) is given by matrix multiplication: $L[\vec{v}] = \mathbf{A}^{m \times n} \vec{v}$. (!?!)

Proof: Key idea: what does linear function do to basis vectors? Let $\hat{e}_1, \dots, \hat{e}_n$ be standard basis of \mathbb{R}^n , and let

$\hat{e}'_1, \dots, \hat{e}'_m$ be standard basis of \mathbb{R}^m . Since $L[\hat{e}_j] \in \mathbb{R}^m$, we can write it as a linear combination of the latter basis vectors:

$$L[\hat{e}_j] = \vec{a}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = a_{1j}\hat{e}'_1 + \dots + a_{mj}\hat{e}'_m, \quad j = 1, \dots, n. \quad (**)$$

Let's construct: $\mathbf{A}^{m \times n} = (\vec{a}_1 \dots \vec{a}_n) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, Whose columns are the image vectors (**).

Using (*), we then compute the effect of L on a general vector $\vec{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$:

$$L[\vec{v}] = L[v_1\hat{e}_1 + \dots + v_n\hat{e}_n] = v_1L[\hat{e}_1] + \dots + v_nL[\hat{e}_n] = v_1\vec{a}_1 + \dots + v_n\vec{a}_n = \mathbf{A}\vec{v}.$$

The final inequality follows from our basic formula: $\mathbf{A}\vec{c} = c_1\vec{a}_1 + \dots + c_k\vec{a}_k$.

We conclude that the vector $L[\vec{v}]$ coincides with the vector $\mathbf{A}\vec{v}$. ■

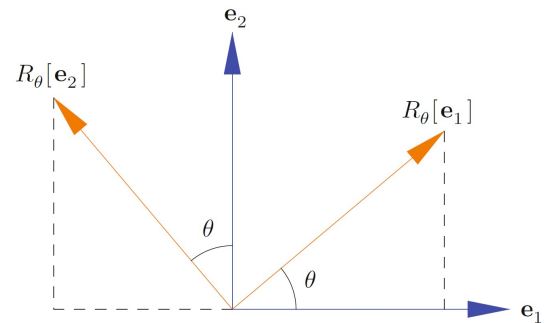
This proof shows how to construct the matrix representative of a linear function L . That is: $\mathbf{A} = [L[\hat{e}_1] \dots L[\hat{e}_n]]$.

Example: Rotations! Let's find matrix representative of rotating 2D space by θ .

According to the proof above, we want:

$$L[\vec{v}] = \mathbf{A}\vec{v} = [L[\hat{e}_1] \quad L[\hat{e}_2]]\vec{v}.$$

And note that rotating \hat{e}_1 by θ gives $(\cos\theta)\hat{e}_1 + (\sin\theta)\hat{e}_2 = (\cos\theta, \sin\theta)$.



rotating \hat{e}_2 by θ gives $-(\sin\theta)\hat{e}_1 + (\cos\theta)\hat{e}_2 = (-\sin\theta, \cos\theta)$.

$$\text{Therefore: } \mathbf{A}\vec{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{v}.$$

Example: Explain why $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $\vec{F}(\vec{x}) = \begin{bmatrix} x + e^y \\ 2x + y \end{bmatrix}$ is not linear.

$$\vec{F}(\vec{0}) = \begin{bmatrix} 0 + e^0 \\ 2(0) + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{F}(c\vec{x}) = \begin{bmatrix} cx + e^{cy} \\ 2cx + cy \end{bmatrix} \neq \begin{bmatrix} cx + ce^y \\ 2cx + cy \end{bmatrix} = c\vec{F}(\vec{x}).$$

Example: Explain why the translation function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by

$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$ for $a, b \in \mathbb{R}$, is almost never linear. Precisely when is it linear?

$$T \left(c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = T \left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix} \right) = \begin{bmatrix} c_1x_1 + c_2x_2 + a \\ c_1y_1 + c_2y_2 + b \end{bmatrix} \neq \begin{bmatrix} c_1x_1 + c_2x_2 + 2a \\ c_1y_1 + c_2y_2 + 2b \end{bmatrix}$$

$$= \begin{bmatrix} c_1x_1 + a \\ c_1y_1 + b \end{bmatrix} + \begin{bmatrix} c_2x_2 + a \\ c_2y_2 + b \end{bmatrix} = c_1 T \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2 T \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad (\text{unless } a = b = 0).$$

Linear Operators

Functions spaces: Much wider variety of linear operators available (not just matrices). Complete classification is out of the question. But let's look at some common examples:

Example: Evaluation of a Function: For $f \in C^0[a, b]$ operate as $L[f] = f(x_0)$ with $L : C^0[a, b] \rightarrow \mathbb{R}$.

$$L[cf + dg] = (cf + dg)(x_0) = cf(x_0) + dg(x_0) = cL[f] + dL[g].$$

Example: Integration operator on $f \in C^0[a, b]$ as $\mathfrak{I}[f] = \int_a^b f(x)dx$ with $\mathfrak{I} : C^0[a, b] \rightarrow \mathbb{R}$.

$$\mathfrak{I}[cf + dg] = \int_a^b (cf(x) + dg(x))dx = c \int_a^b f(x)dx + d \int_a^b g(x)dx = c\mathfrak{I}[f] + d\mathfrak{I}[g].$$

Example: Indefinite Integral operator on $f \in C^0[a, b]$ as $J[f] = \int_a^x f(x)dx$ with $J : C^0[a, b] \rightarrow C^0[a, b]$.

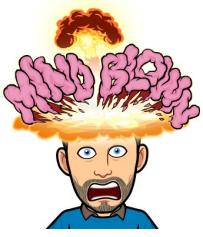
$$J[cf + dg] = \int_a^x (cf(x) + dg(x))dx = c \int_a^x f(x)dx + d \int_a^x g(x)dx = cJ[f] + dJ[g].$$

Example: Differentiation operator on $f \in C^1[a, b]$ as $D[f] = f'$ with $D : C^1[a, b] \rightarrow C^0[a, b]$.

$$D[cf + dg] = (cf(x) + dg(x))' = cf'(x) + dg'(x) = cD[f] + dD[g].$$

The Space of Linear Functions

Let's take linearity up a notch! Given two vector spaces V, W , let's use $\mathcal{L}(V, W)$ to denote set of all linear functions $L : V \rightarrow W$.



It turns out $\mathcal{L}(V, W)$ (in addition to operating on vector spaces) is *itself*, a vector space!

Addition Notation/Definition: $(L + M)[\vec{v}] := L[\vec{v}] + M[\vec{v}]$.

So is $L + M$ linear? $(L + M)[a\vec{v} + b\vec{w}] = L[a\vec{v} + b\vec{w}] + M[a\vec{v} + b\vec{w}]$ (definition above)

$$= aL[\vec{v}] + bL[\vec{w}] + aM[\vec{v}] + bM[\vec{w}] \quad (\text{linearity of } L, M)$$

$$= (aL[\vec{v}] + aM[\vec{v}]) + (bL[\vec{w}] + bM[\vec{w}])$$

$$= a(L + M)[\vec{v}] + b(L + M)[\vec{w}]. \quad (\text{definition above}) \quad \text{So, yes. } L + M \text{ is linear.}$$

Scalar Mult. Notation/Definition: $(cL)[\vec{v}] := cL[\vec{v}]$. Linearity shown similarly to above.

Zero Function: $O[\vec{v}] := \vec{0}$. Vector field axioms are satisfied with these definitions (check them!).

Concretely: Space of linear transformations in the plane: $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is identified with space $\mathcal{M}_{2 \times 2}$ of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\text{Standard basis: } \mathbf{E}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{E}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{E}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{E}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\text{Every matrix uniquely written as: } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\mathbf{E}_{11} + b\mathbf{E}_{12} + c\mathbf{E}_{21} + d\mathbf{E}_{22}.$$

Dual Spaces

Definition: The *dual space* to a vector space V is the vector space $V^* = \mathcal{L}(V, \mathbb{R})$ consisting of all real valued linear functions $\ell : V \rightarrow \mathbb{R}$.

If $V = \mathbb{R}^n$, then, by previous them, every linear function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by multiplication by $1 \times n$ matrix, i.e., a row vector.

$$\ell[\vec{v}] = \vec{a}^T \vec{v} = a_1 v_1 + \dots + a_n v_n.$$

Therefore, we can identify the dual space $(\mathbb{R}^n)^*$ with the space of row vectors with n entries.

Row vectors should more properly be viewed as real valued linear functions, the dual objects to column vectors.

Theorem: Let V be a finite dimensional real inner product space. Then every linear function $\ell : V \rightarrow \mathbb{R}$ is given by taking the inner product with a fixed vector $\vec{a} \in V : \ell[\vec{v}] = \langle \vec{a}, \vec{v} \rangle$.

Proof: Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis of V . If we write $\vec{v} = y_1 \vec{v}_1 + \dots + y_n \vec{v}_n$, then, by linearity,

$$\ell[\vec{v}] = y_1 \ell[\vec{v}_1] + \dots + y_n \ell[\vec{v}_n] = b_1 y_1 + \dots + b_n y_n, \text{ where } b_i = \ell[\vec{v}_i]. \quad (*)$$

On the other hand, if we write $\vec{a} = x_1 \vec{v}_1 + \dots + x_n \vec{v}_n$, then

$$\langle \vec{a}, \vec{v} \rangle = \sum_{i,j=1}^n x_j y_i \langle \vec{v}_i, \vec{v}_j \rangle = \sum_{i,j=1}^n g_{ij} x_j y_i, \quad (**)$$

where $G = (g_{ij})$ is the $n \times n$ Gram matrix with entries $g_{ij} = \langle \vec{v}_i, \vec{v}_j \rangle$.

Equality of $(*)$ and $(**)$ requires that $G\vec{x} = \vec{b}$, where $\vec{x} = (x_1, \dots, x_n)^T$, $\vec{b} = (b_1, \dots, b_n)^T$.

Invertibility of G is guaranteed by Theorem 3.34 allows us to solve for $\vec{x} = G^{-1}\vec{b}$ and thereby construct the desired \vec{a} .

In particular, if $\vec{v}_1, \dots, \vec{v}_n$ is an orthonormal basis, then $G = 1$ and hence $\vec{a} = b_1 \vec{v}_1 + \dots + b_n \vec{v}_n$. ■

Example: Write down a basis for, and dimension of, the linear function space $\mathcal{L}(P^3, \mathbb{R})$.

This is 4D. In particular, if you view the polynomials as vectors in \mathbb{R}^4 , the polynomial $a_3 x^3 + a_2 x^2 + a_1 x + a_0$ can be written (a_3, a_2, a_1, a_0) , with basis $L_0 = (0, 0, 0, 1)^T$, $L_1 = (0, 0, 1, 0)^T$, etc.

Example: Given a basis $\vec{v}_1, \dots, \vec{v}_n$ of V , let the dual basis ℓ_1, \dots, ℓ_n of V^* consists of the linear functions uniquely defined by

the requirements $\ell_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$ Find the dual basis for: $\vec{v}_1 = (1, 1, 0)$, $\vec{v}_2 = (1, 0, 1)$, and $\vec{v}_3 = (0, 1, 1)$.

Note we were given column vectors. However, we are looking for elements ℓ_i of $(\mathbb{R}^3)^*$ (row vectors) such that

$$\ell_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let's require $\ell_1(\vec{v}_1) = \ell_1((1, 1, 0)) = (\ell_{11}, \ell_{12}, \ell_{13})^T (1, 1, 0) = 1$.

And similarly that $(\ell_{11}, \ell_{12}, \ell_{13})^T (1, 0, 1) = (\ell_{11}, \ell_{12}, \ell_{13})^T (0, 1, 1) = 0$.

From this we get: $\ell_{11} + \ell_{12} = 1$, $\ell_{11} + \ell_{13} = 0$, $\ell_{12} + \ell_{13} = 0$.

⇒ From the 2nd two eqs: $\ell_{11} = -\ell_{13}$, $\ell_{12} = -\ell_{13}$, so $\ell_{11} = \ell_{12} = -\ell_{13}$.

⇒ From the 1st eq: $\ell_{11} + \ell_{12} = 2\ell_{11} = 1$ or $\ell_{11} = \frac{1}{2}$. Therefore: $(\ell_{11}, \ell_{12}, \ell_{13}) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

Let's require $\ell_2(\vec{v}_2) = \ell_2((1, 0, 1)) = (\ell_{21}, \ell_{22}, \ell_{23})^T(1, 0, 1) = 1$.

And similarly that $(\ell_{21}, \ell_{22}, \ell_{23})^T(1, 1, 0) = (\ell_{21}, \ell_{22}, \ell_{23})^T(0, 1, 1) = 0$.

From this we get: $\ell_{21} + \ell_{23} = 1$, $\ell_{21} + \ell_{22} = 0$, $\ell_{22} + \ell_{23} = 0$.

⇒ $\ell_{21} = \ell_{23} = -\ell_{22}$ and $\ell_{21} = \frac{1}{2}$. Therefore: $(\ell_{21}, \ell_{22}, \ell_{23}) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$.

Similarly, we find $(\ell_{31}, \ell_{32}, \ell_{33}) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

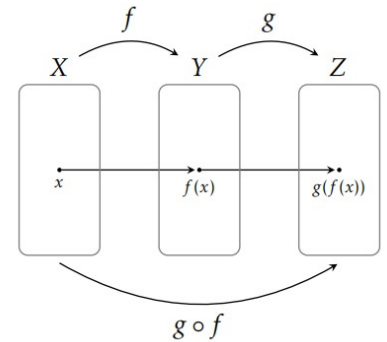
Therefore, the dual basis is $\{(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$.

Composition

So far, we've added linear functions, and multiplied them by scalars.

What about composing them?

Lemma: Let V, W, Z be vector spaces. If $L : V \rightarrow W$ and $M : W \rightarrow Z$ are linear functions, then the *composite function* $M \circ L : V \rightarrow Z$, defined by $(M \circ L)[\vec{v}] := M[L[\vec{v}]]$ is also linear.



Proof: $(M \circ L)[c\vec{v} + d\vec{w}] = M[L[c\vec{v} + d\vec{w}]]$ (definition)

$$= M[cL[\vec{v}] + dL[\vec{w}]] \quad (\text{linearity of } L)$$

$$= cM[L[\vec{v}]] + dM[L[\vec{w}]] \quad (\text{linearity of } M)$$

$$= c(M \circ L)[\vec{v}] + d(M \circ L)[\vec{w}] \quad (\text{definition}) \quad \blacksquare$$

Concretely: The *commutator* of two linear transformations $L, M : V \rightarrow V$ on a vector space V is:

$K := [L, M] := L \circ M - M \circ L$, where $[L, M]$ is referred to as a *Lie bracket*.

Prove that the commutator K is a linear transformation on V .

$$K[a\vec{v} + b\vec{w}] = [L, M][a\vec{v} + b\vec{w}] = (L \circ M - M \circ L)[a\vec{v} + b\vec{w}] \quad (\text{definition of } K)$$

$$= L[M[a\vec{v} + b\vec{w}]] - M[L[a\vec{v} + b\vec{w}]] \quad (\text{definition of composition})$$

$$= L[aM[\vec{v}] + bM[\vec{w}]] - M[aL[\vec{v}] - bL[\vec{w}]] \quad (\text{linearity of } L, M)$$

$$= aL[M[\vec{v}]] + bL[M[\vec{w}]] - aM[L[\vec{v}]] - bM[L[\vec{w}]] \quad (\text{linearity of } L, M)$$

$$= (aL[M[\vec{v}]] - aM[L[\vec{v}]]) + (bL[M[\vec{w}]] - bM[L[\vec{w}]]) \quad (\text{algebra})$$

$$= a(L \circ M[\vec{v}] - M \circ L[\vec{v}]) + b(L \circ M[\vec{w}] - M \circ L[\vec{w}]) \quad (\text{definition of composition})$$

$$= a(L \circ M - M \circ L)[\vec{v}] + b(L \circ M - M \circ L)[\vec{w}] = a[L, M][\vec{v}] + b[L, M][\vec{w}] \quad (\text{definition of } K)$$

$$= aK[\vec{v}] + bK[\vec{w}]. \quad (\text{definition of } K)$$

The Jacoby Identity: $[[L, M], N] + [[N, L], M] + [[M, N], L] = O$.

Applications: **Cross product** $\vec{a} \times \vec{b}$ and **Lie bracket** operation $[L, M]$ satisfy the Jacobi identity. In **analytical mechanics**, the Jacobi identity is satisfied by the **Poisson brackets** $\{f, g\}$. In **quantum mechanics**, it is satisfied by operator commutators on a Hilbert space and equivalently in the **phase space** formulation of quantum mechanics by the **Moyal bracket** $\{\{f, g\}\}$.

Prove that the Jacoby Identity: $[[L, M], N] + [[N, L], M] + [[M, N], L] = O$ is valid for any three transformations.

$$\begin{aligned} [[L, M], N] &= [L, M] \circ N - N \circ [L, M] = (L \circ M - M \circ L) \circ N - N \circ (L \circ M - M \circ L) \\ &= (L \circ M \circ N - M \circ L \circ N) - (N \circ L \circ M - N \circ M \circ L) = L \circ M \circ N - M \circ L \circ N - N \circ L \circ M + N \circ M \circ L \end{aligned}$$

$$\begin{aligned} \text{Similarly: } [[N, L], M] &= N \circ L \circ M - L \circ N \circ M - M \circ N \circ L + M \circ L \circ N, \\ \text{and } [[M, N], L] &= M \circ N \circ L - N \circ M \circ L - L \circ M \circ N + L \circ N \circ M. \end{aligned}$$

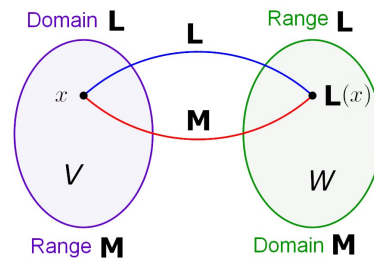
$$\begin{aligned} \text{Therefore: } & [[L, M], N] + [[N, L], M] + [[M, N], L] \\ &= (L \circ M \circ N - M \circ L \circ N - N \circ L \circ M + N \circ M \circ L) + (N \circ L \circ M - L \circ N \circ M - M \circ N \circ L + M \circ L \circ N) \\ & \quad + (M \circ N \circ L - N \circ M \circ L - L \circ M \circ N + L \circ N \circ M) \\ &= (L \circ M \circ N - L \circ M \circ N + L \circ N \circ M - L \circ N \circ M) + (N \circ L \circ M - N \circ L \circ M + N \circ M \circ L - N \circ M \circ L) \\ & \quad + (M \circ N \circ L - M \circ N \circ L + M \circ L \circ N - M \circ L \circ N) \\ &= O. \end{aligned}$$

■

Inverses

Do linear functions have inverses? We've seen that square matrices do.

More generally, we have the following:



Definition: Let $L : V \rightarrow W$ be a linear function. If $M : W \rightarrow V$ is a function such that:

$$L \circ M = I_W, \quad M \circ L = I_V, \quad (***)$$

are equal to the identity function (in their respective spaces), then we call M the *inverse* of L and write $M = L^{-1}$.

In other words, being an inverse requires: $L[M[\vec{w}]] = \vec{w}$ for all $\vec{w} \in W$, and $M[L[\vec{v}]] = \vec{v}$ for all $\vec{v} \in V$.

When it exists, the inverse is unique (proof part of book exercise)!

Informally: Observe that since we relabeled the right- and left-inverse (M) of L as L^{-1} in $(***)$, then $(L^{-1})^{-1} = L$.

Lemma: If it exists, the inverse of a linear function is also a linear function.

Proof: Let L, M satisfy the conditions of the previous definition.

Given $\vec{w}, \vec{w}' \in W$. We wish to show that M is linear, or (given scalars c, d) that $M[c\vec{w} + d\vec{w}'] = cM[\vec{w}] + dM[\vec{w}']$.

Since M and L are inverses, we note $\vec{w} = (L \circ M)[\vec{w}] = L[M[\vec{w}]]$,

and $\vec{w}' = (L \circ M)[\vec{w}'] = L[M[\vec{w}']]$.

Therefore, using only linearity of L :

$$M[c\vec{w} + d\vec{w}'] = M[cL[M[\vec{w}]] + dL[M[\vec{w}']]] \quad (\text{from above})$$

$$= M[L[cM[\vec{w}] + dM[\vec{w}']]] \quad (\text{linearity of } L)$$

$$= (M \circ L)[cM[\vec{w}] + dM[\vec{w}']] \quad (\text{definition of composition})$$

$$= cM[\vec{w}] + dM[\vec{w}'], \quad (M \text{ and } L \text{ are inverses})$$

proving linearity of M . ■

On finite dimensional domains, since linear transformations can be written as matrices, this results in a very simple situation.

That is, recall that if $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, such that L, M are given by matrix multiplication by \mathbf{A} and \mathbf{B} respectively, then the inverse definition's conditions ($*$ $*$ $*$) reduce to the usual conditions: $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$.

In particular, \mathbf{I} is an $n \times n$ matrix, requiring $m = n$, and for its coefficient matrix \mathbf{A} to be nonsingular.

However, on infinite dimensional function spaces, things are more subtle.

Concretely: Let $D[f] = f'$ represent differentiation on $C^1[a, b]$, and

$$J[f] = g \text{ represent integration, where } g(x) = \int_a^x f(y) dy \text{ on } C^0[a, b].$$

We often consider these "inverse" operations.

Indeed, observe that $(D \circ J)[f] = D[J[f]] = D[g] = g' = f$, since

$$g'(x) = \frac{d}{dx} \int_a^x f(y) dy = f(x). \quad (\text{Fund. Thm. Calc})$$

However, what about $(J \circ D)[f]$? This is $J[D[f]] = J[f'] = h$, where

$$h(x) = \int_a^x f'(y) dy = f(x) - f(a).$$

So $h(x) \neq f(x)$, unless $f(a) = 0$. So, $J \circ D \neq I_{C^1[a, b]}$.

In other words, D is a left-inverse for J , but not a right-inverse!

If we restrict D to $V := \{f : f(a) = 0\} \subset C^1[a, b]$, then an inverse is defined (as seen above).

But notice that $V \subsetneq C^1[a, b] \subsetneq C^0[a, b]$.

So J defines a one-to-one invertible, linear map from a vector space $C^0[a, b]$ to a proper subspace $V \subsetneq C^0[a, b]$!

This can't happen in finite dimensions. A matrix (linear map) is invertible only when the size of it's image is the same as it's domain and codomain.

Concretely: Determine if linear function $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by reflection through the x -axis has an inverse. If so, describe it.

How does this transformation $L\langle x, y \rangle \rightarrow \langle x, -y \rangle$ affect the basis vectors?

...

$$L\langle 1, 0 \rangle \rightarrow \langle 1, 0 \rangle, \quad L\langle 0, 1 \rangle \rightarrow \langle 0, -1 \rangle.$$

$$\text{Therefore: } L\langle x, y \rangle = \mathbf{A}\vec{x} = [L(\hat{e}_1) \ L(\hat{e}_2)]\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{x}.$$

$$\text{And observe that } \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{A}.$$

So, yes L has an inverse. And it is its *own* inverse (Which intuitively/geometrically makes sense)!

