

Example: $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 6xy + y^2$ and $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 6xy + y^2$.

$p(\vec{x})$ is more general than quadratic form: has linear & constant terms.

Rewrite as: $p(\vec{x}) = \vec{x}^T \mathbf{K} \vec{x} - 2\vec{x}^T \vec{f} + c$, where $\mathbf{K} = (k_{ij})$ and $\vec{f} = (f_i)$.

Example: $p(\vec{x}) = 4x_1^2 - 2x_1x_2 + 3x_2^2 + 3x_1 - 2x_2 + 1$.

Rewritten: $p(\vec{x}) = \vec{x}^T \mathbf{K} \vec{x} - 2\vec{x}^T \vec{f} + c = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} + 1$.

In multi-var version, instead of requiring $a > 0$ (as we had in single-var case), we need $\mathbf{K} > 0$ in order to obtain a *unique* minimum.

Theorem: If $\mathbf{K} > 0$ (and hence symmetric), then quadratic function $p(\vec{x}) = \vec{x}^T \mathbf{K} \vec{x} - 2\vec{x}^T \vec{f} + c$

has a unique minimizer, which is the solution to the linear system $\mathbf{K} \vec{x} = \vec{f}$, namely $\vec{x}^* = \mathbf{K}^{-1} \vec{f}$.

The minimum value of $p(\vec{x})$ is equal to any of the following expressions:

$$p(\vec{x}^*) = p(\mathbf{K}^{-1} \vec{f}) = c - \vec{f}^T \mathbf{K}^{-1} \vec{f} = c - \vec{f}^T \vec{x}^* = c - (\vec{x}^*)^T \mathbf{K} \vec{x}^*. \quad (*)$$

Proof: Recall positive definiteness implies \mathbf{K} is nonsingular, hence $\mathbf{K} \vec{x} = \vec{f}$ has unique solution $\vec{x}^* = \mathbf{K}^{-1} \vec{f}$.

But is it the unique minimizer of $p(\vec{x})$?

Well, $\forall \vec{x} \in \mathbb{R}^n$, if $\vec{f} = \mathbf{K} \vec{x}^*$, it follows that

$$p(\vec{x}) = \vec{x}^T \mathbf{K} \vec{x} - 2\vec{x}^T \vec{f} + c$$

$$= \vec{x}^T \mathbf{K} \vec{x} - 2\vec{x}^T \mathbf{K} \vec{x}^* + c$$

$$= \vec{x}^T \mathbf{K} \vec{x} - \vec{x}^T \mathbf{K} \vec{x}^* - (\vec{x}^T \mathbf{K} \vec{x}^*)^T + c \quad (\text{since the transpose of a constant is itself})$$

$$= \vec{x}^T \mathbf{K} \vec{x} - \vec{x}^T \mathbf{K} \vec{x}^* - (\vec{x}^*)^T \mathbf{K} \vec{x} + (\vec{x}^*)^T \mathbf{K} \vec{x}^* + [c - (\vec{x}^*)^T \mathbf{K} \vec{x}^*] \quad (\text{distributed transpose, and added/subtracted same term})$$

$$= \vec{x}^T \mathbf{K} (\vec{x} - \vec{x}^*) - (\vec{x}^*)^T \mathbf{K} (\vec{x} - \vec{x}^*) + [c - (\vec{x}^*)^T \mathbf{K} \vec{x}^*] \quad (\text{factoring out})$$

$$= (\vec{x} - \vec{x}^*)^T \mathbf{K}(\vec{x} - \vec{x}^*) + [c - (\vec{x}^*)^T \mathbf{K} \vec{x}^*], \quad (\text{factoring out again}) \quad (**)$$

The first term in (**) has form $\vec{y}^T \mathbf{K} \vec{y}$, where $\vec{y} = \vec{x} - \vec{x}^*$.

Since we assumed \mathbf{K} is positive definite, we know $\vec{y}^T \mathbf{K} \vec{y} > 0$ for all $\vec{y} \neq \vec{0}$.

Thus, the first term achieves its minimum value (namely zero) **iff** $\vec{0} = \vec{y} = \vec{x} - \vec{x}^*$.

Since \vec{x}^* is fixed, the second, bracketed term in (**) doesn't depend on \vec{x} , and hence the minimizer of $p(\vec{x})$ coincides with the minimizer of the first term, namely $\vec{x} = \vec{x}^*$.

Moreover, the minimum value of $p(\vec{x})$ is equal to the constant term: $p(\vec{x}^*) = c - (\vec{x}^*)^T \mathbf{K} \vec{x}^*$.

The alternative expressions in (*) follow from simple substitutions. ■

Continued Example: Find minimizer of

$$p(\vec{x}) = \vec{x}^T \mathbf{K} \vec{x} - 2\vec{x}^T \vec{f} + c = [x_1 \ x_2] \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2[x_1 \ x_2] \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} + 1.$$

According to the thm, we must solve: $\mathbf{K} \vec{x} = \vec{f}$, which from above is:

$$\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 4 & -1 & -\frac{3}{2} \\ -1 & 3 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 4 & -1 & -\frac{3}{2} \\ 0 & \frac{11}{4} & \frac{5}{8} \end{array} \right].$$

\mathbf{K} appears regular, with positive pivots. Therefore: $\mathbf{K} > 0$ and $p(\vec{x})$ does have a global minimum.

Solving yields minimizer is: $\vec{x}^* = \begin{bmatrix} -\frac{7}{22} \\ \frac{5}{22} \end{bmatrix}$.

Therefore the minimum (most easily acquired from $p(\vec{x}^*) = c - \vec{f}^T \vec{x}^*$ from the thm) is

$$p(\vec{x}^*) = p\left(-\frac{7}{22}, \frac{5}{22}\right) = 1 - \left[-\frac{3}{2} \ 1\right] \begin{bmatrix} -\frac{7}{22} \\ \frac{5}{22} \end{bmatrix} = \frac{13}{44}.$$

Example: For $\mathbf{K} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, $\vec{f} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, and $c = -3$, write out the quadratic function $p(\vec{x}) = \vec{x}^T \mathbf{K} \vec{x} - 2\vec{x}^T \vec{f} + c$,

where $\vec{x} \in \mathbb{R}^n$. Then either find the minimizer \vec{x}^* and minimum value $p(\vec{x}^*)$, or explain why there is none.

According to the thm, we must solve: $\mathbf{K}\vec{x} = \vec{f}$, which from above is:

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 3 & -2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & \frac{5}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 2 & -2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & \frac{5}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 12 & -7 \end{array} \right].$$

\mathbf{K} was regular, with positive pivots. Therefore: $\mathbf{K} > 0$ and $p(\vec{x})$ does have a global minimum.

Solving for minimizer: $\vec{x}^* = \begin{bmatrix} \frac{7}{12} \\ -\frac{1}{6} \\ -\frac{11}{12} \end{bmatrix}.$

Therefore the minimum (most easily acquired from $p(\vec{x}^*) = c - \vec{f}^T \vec{x}^*$ from the thm) is

$$p(\vec{x}^*) = p\left(-\frac{7}{22}, \frac{5}{22}\right) = -3 - [1 \ 0 \ -2] \begin{bmatrix} \frac{7}{12} \\ -\frac{1}{6} \\ -\frac{11}{12} \end{bmatrix} = -\frac{65}{12}.$$