### 3.5 Completing the Square

## Determining the Positive Definiteness of a Matrix

"Completing the square" has previously assisted you in deriving the quadratic formula, and later for integrating various types of rational and algebraic functions.

Given: $q(x) = ax^2 + 2bx + c = 0$

$$q(x) = a(x + \tfrac{b}{a})^2 + \tfrac{ac-b^2}{a} = 0.$$

As a result: $(x + \frac{b}{a})^2 = \frac{b^2-ac}{a^2}$ and $x = \frac{-b \pm \sqrt{b^2-ac}}{a}$.

Similarly: given $q(x_1,x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, where $a \neq 0$,

$$q(x_1,x_2) = a(x_1 + \tfrac{b}{a}x_2)^2 + \tfrac{ac-b^2}{a}x_2^2$$

$$= ay_1^2 + \tfrac{ac-b^2}{a}y_2^2, \qquad (1)$$

where $y_1 = x_1 + \frac{b}{a}x_2$ and $y_2 = x_2$. (2)

Expression (1) is positive definite in $y_1, y_2$ if $a > 0$ and $\frac{ac-b^2}{a} > 0$.

This would mean, $q(x_1,x_2) \geq 0$. We get equality **iff**

$$y_1 = y_2 = 0 \qquad \Leftrightarrow \qquad x_1 = x_2 = 0.$$

**Goal**: **Generalize to More Variables**

Observe quadratic form $q(x_1,x_2)$ above can be rewriten as: $\vec{x}^T\mathbf{K}\vec{x}$, where $\mathbf{K} =$ ??

$$\mathbf{K} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad \vec{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

RHS of (1) can be written as $\hat{q}(\vec{y}) := \vec{y}^T\mathbf{D}\vec{y}$, (3) where $\mathbf{D} := \begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}$, $\vec{y} := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

From (2), we can write $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + \frac{b}{a}x_2 \\ x_2 \end{bmatrix}$ or $\vec{y} = \mathbf{L}^T\vec{x}$ where $\mathbf{L}^T := \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$.

Substituting this into (3), we obtain: $\vec{y}^T\mathbf{D}\vec{y} = (\mathbf{L}^T\vec{x})^T\mathbf{D}(\mathbf{L}^T\vec{x})$

$$= \vec{x}^T\mathbf{LDL}^T\vec{x} = \vec{x}^T\mathbf{K}\vec{x}, \text{ where } \mathbf{K} = \mathbf{LDL}^T. \qquad (4)$$

In other words, given $q(\vec{x})$ (and therefore **K**), you can complete the square by calculating the $\mathbf{LDL}^T$ decomposition of **K**.

($D$ gives you the coefficients, and $L$ gives you the squared quantities of (1))

A previous theorem says that regular symmetric matrices are precisely those that admit an $\mathbf{LDL}^T$ decomposition. Therefore (4) is valid for all regular symmetric matrices.

This allows us to write a quadratic form as a sum of squares: $q(\vec{x}) = \vec{x}^T\mathbf{K}\vec{x} = \vec{y}^T\mathbf{D}\vec{y} = d_1y_1^2 + \ldots + d_ny_n^2$, where $\vec{y} = \mathbf{L}^T\vec{x}$.

The $d_i$ are the diagonal entries of **D**, the pivots of **K**. So:

Regular & Symmetric $\Rightarrow$ $\mathbf{LDL}^T$ factorable $\Rightarrow$ "Complete the square-able"

"Complete the square-able" and $d_i > 0$ $\Rightarrow$ Pos. def.

## **How about with more variable**?

**Theorem**: Given a symmetric $\mathbf{K}^{n\times n}$, it is positive definite **iff** it is regular and has all positive pivots.

**Proof**: If upper left $k_{11}$ (first pivot) is not strictly positive, **K** cannot be positive definite because $q(\vec{e}_1) = \vec{e}_1^T\mathbf{K}\vec{e}_1 = k_{11} \leq 0$.

Otherwise, suppose $k_{11} > 0$. We can write: $q(\vec{x}) = \vec{x}^T\mathbf{K}\vec{x}$

$$= (k_{11}x_1^2 + 2k_{12}x_1x_2 + \ldots + 2k_{1n}x_1x_n) + \ldots + (k_{22}x_2^2 + 2k_{23}x_2x_3 + \ldots + 2k_{2n}x_2x_n) + \ldots + (2k_{1n}x_1x_n + \ldots + k_{nn}x_n^2)$$

$$= k_{11}\left(x_1 + \frac{k_{12}}{k_{11}}x_2 + \ldots + \frac{k_{1n}}{k_{11}}x_n\right)^2 + \tilde{q}(x_2, \ldots, x_n)$$

$$= k_{11}(x_1 + \ell_{21}x_2 + \ldots + \ell_{n1}x_n)^2 + \tilde{q}(x_2, \ldots, x_n). \qquad (5)$$

Claim: $q(\vec{x})$ is positive definite **iff** $\widetilde{q}$ is positive definite. (must show both directions $\Leftarrow$ & $\Rightarrow$)

$\Leftarrow$ Indeed, if $\widetilde{q}$ is positive definite and $k_{11} > 0$, then $q(\vec{x})$ is the sum of two positive quantities, which simultaneously vanish **iff** $x_1 = x_2 = \ldots = x_n = 0$.

$\Rightarrow$ Showing contrapositive, suppose $\widetilde{q}(x_2^*, \ldots, x_n^*) \leq 0$ for some $x_2^*, \ldots, x_n^*$, not all zero.

Setting $x_1^* = -\ell_{21}x_2^* - \ldots - \ell_{n1}x_n^*$ makes the initial square term in (5) equal to zero,
so $q(x_2^*, \ldots, x_n^*) = \widetilde{q}(x_2^*, \ldots, x_n^*) \leq 0$. ■

Cholesky Factorization is cool, but I won't test you on it.