

Applied Linear Algebra

Textbook: *Applied Linear Algebra* by Olver and Shakiban

2.3 Span and Linear Independence

Definition: Let $\vec{v}_1, \dots, \vec{v}_k$ be elements of a vector space V . A sum the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \sum_{i=1}^k c_i\vec{v}_i$, where the coefficients c_1, \dots, c_k are any scalars, is known as a *linear combination* of the elements $\vec{v}_1, \dots, \vec{v}_k$. Their span is the subset $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\} \subset V$ consisting of all possible linear combinations with scalars $c_1, \dots, c_k \in \mathbb{R}$.

Proposition: The span $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ of any finite collection of vector space elements $\vec{v}_1, \dots, \vec{v}_k \in V$ is a subspace of the underlying vector space V .

Proof: Recall $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_m\} = \{c_1\vec{v}_1 + \dots + c_m\vec{v}_m \mid c_i \in \mathbb{R}\}$.

a) $\vec{0} \in W$?

Set $c_1 = c_2 = \dots = c_m = 0$. Then,

$$c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}.$$

b) Closed under addition?

Let $\vec{a}, \vec{b} \in W$. Then, there exist constants a_i, b_i such that:

$$\vec{a} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m,$$

$$\vec{b} = b_1\vec{v}_1 + \dots + b_m\vec{v}_m.$$

$$\text{Therefore: } \vec{a} + \vec{b} = (a_1 + b_1)\vec{v}_1 + \dots + (a_m + b_m)\vec{v}_m \in W.$$

c) Closed under scalar multiplication?

Definition: The vector space elements $\vec{v}_1, \dots, \vec{v}_k \in V$ are called linearly dependent if there exists scalars c_1, \dots, c_k , *not all zero*, such that $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$. Elements that are not linearly dependent are called linearly independent.

If $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent, it doesn't mean every \vec{v}_i can be written as a combination of the others.

It implies at least one \vec{v}_i can.

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix} \right\} \dots$

$$7\vec{v}_1 + 5\vec{v}_2 + 0\vec{v}_3 = \vec{v}_4. \quad \text{However, } \vec{v}_3 \text{ can't be written in terms of the others.}$$

Thm: Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ and let $\mathbf{A}^{n \times k} = (\vec{v}_1 \dots \vec{v}_k)$ be the corresponding matrix whose columns are the given vectors.

- ◆ Vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ are linearly dependent **iff** there is a nonzero solution $\vec{c} \neq \vec{0}$ to the homogeneous $\mathbf{A}\vec{c} = \vec{0}$.
- ◆ Vectors are linearly independent **iff** the only solution to the homogeneous system $\mathbf{A}\vec{c} = \vec{0}$ is the trivial one, $\vec{c} = \vec{0}$.
- ◆ Vector \vec{b} lies in the span of $\vec{v}_1, \dots, \vec{v}_k$ **iff** the linear system $\mathbf{A}\vec{c} = \vec{b}$ is compatible, i.e., has at least one solution.

We prove the first statement, leaving the other two as exercises for you.

Proof: The condition that $\vec{v}_1, \dots, \vec{v}_k$ be linearly dependent is that there exists a nonzero vector

$$\vec{c} = (c_1, c_2, \dots, c_k)^T \neq \vec{0} \text{ such that } c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}.$$

But recall that $\mathbf{A}\vec{c} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$.

Therefore, linear dependence requires the existence of a nontrivial solution to $\mathbf{A}\vec{c} = \vec{0}$. ■

Lemma: Any collection of $k > n$ vectors in \mathbb{R}^n is linearly dependent.

Proposition: A set of k vectors in \mathbb{R}^n is linearly independent **iff** the corresponding $n \times k$ matrix \mathbf{A} has rank k .

In particular, this requires $k \leq n$.

Proposition: A collection of k vectors spans \mathbb{R}^n **iff** their $n \times k$ matrix has rank n . In particular, this requires $k \geq n$.

Determining dependence of $\vec{v}_1, \dots, \vec{v}_m$:

Dependent

- ◆ at least one is a linear combination of the others
- ◆ there exists at least one nontrivial "linear relation": $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$.

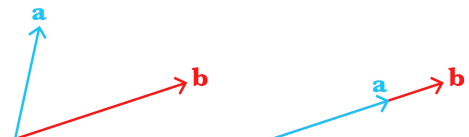
Independent

- ◆ No \vec{v}_i is combination of others
- ◆ The only relation is the trivial one: $0\vec{v}_1 + \dots + 0\vec{v}_m = \vec{0}$.

Examples:

Two vectors in \mathbb{R}^n are linearly...

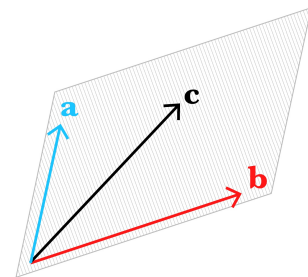
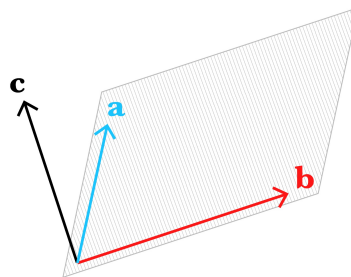
- ◆ Dependent if they're \parallel (one is multiple of other)
- ◆ Independent if not \parallel .



For **three (or more) vectors**, need more than parallel criteria.

◆ None of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ may be parallel to the others, but vectors may still be dependent.

◆ Criteria is whether one is linear combination of other two.



Example: Are $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ independent?

Method 1: Look for possible relations:

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From first coordinate, c_3 must be 0. Second coordinate, c_2 must be 0. Third coordinate, $c_1 = 0$.

Only relation is trivial \Rightarrow independent.

Method 2: $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 5 & 0 \end{array} \right] \xrightarrow{ref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right].$$

Method 3: Compare vectors to check for dependent system.

First vector dependent? One vector can't be dependent.

Ensure second vector not multiple of first.

Now check if third vector is a linear combination of the first two.

$$x \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \text{ obviously not (first component).}$$

Example: Show that the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ are linearly independent.

The corresponding 4×3 matrix is: $\mathbf{A} := \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Must show this has rank 3.

$$\rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Three pivot points imply rank 3.}$$

Which of the following vectors are in their span? $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Keep in mind, the span consists of sums $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$, for any choice of $c_i \in \mathbb{R}$.

So the zero vector $\vec{0}$ can be satisfied when the $c_i = 0$.

For the others, you can attempt to solve the system: $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \vec{c} = \vec{b}$.

Observe that we can attempt to solve them all simultaneously in the following augmented matrix: $[\mathbf{A} \mid \vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 2 & -1 & 1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 3 & -3 & 0 & -2 & 0 \\ 0 & 3 & -3 & 0 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & -3 & -2 \\ 0 & 3 & 0 & 3 & 4 & 3 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 3 & 0 & 3 & 4 & 3 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]. \end{aligned}$$

Observe that \vec{b}_2 has no solution due to the last row.

According to this result, \vec{b}_1 can be formed by: $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = 1\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3.$

And indeed: $1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$

Similarly, \vec{b}_3 can be formed by: $0\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3 = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Suppose (a, b, c, d) lies in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. What conditions must a, b, c, d satisfy?

Since $k_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, we must solve $\begin{bmatrix} 1 & -2 & 2 & | & a \\ 0 & 3 & -2 & | & b \\ 2 & -1 & 1 & | & c \\ 1 & 1 & -1 & | & d \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & -2 & 2 & & a \\ 0 & 3 & -2 & & b \\ 0 & 0 & -3 & 3c - 3b - 6a & \\ 0 & 0 & 0 & -3d + 3c - 3a & \end{bmatrix}.$$

The last row implies that $c = a + d$.

That is the only requirement on the a, b, c, d .

However, you can subsequently determine the requirements on the unique solution determined by this vector:

The 3rd row gives us $k_3 = b + a - d$.

The 2nd row gives us $3k_2 - 2k_3 = b \Rightarrow 3k_2 - 2(b + a - d) = b$

$$\text{or } k_2 = b + \frac{2}{3}a - \frac{2}{3}d.$$

Lastly, from the first row we have: $k_1 - 2k_2 + k_3 = a \Rightarrow k_1 - 2\left(b + \frac{2}{3}a - \frac{2}{3}d\right) + (b + a - d) = a$

$$\Rightarrow k_1 = \frac{4}{3}a + b - \frac{1}{3}d.$$

Therefore, every vector (a, b, c, d) such that $c = a + d$, lies in the span, and specifically:

$$(a, b, c, d) = k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3 = \left(\frac{4}{3}a + b - \frac{1}{3}d\right) \vec{v}_1 + \left(b + \frac{2}{3}a - \frac{2}{3}d\right) \vec{v}_2 + (b + a - d) \vec{v}_3.$$