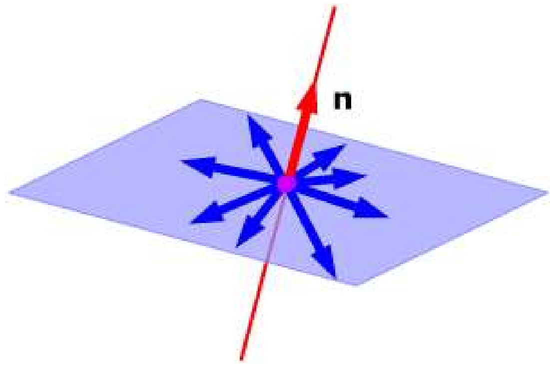
Applied Linear Algebra

Textbook: *Applied Linear Algebra* by Olver and Shakiban

2.2 Subspaces



Subspaces: Given vector space V (e.g., $\mathbb{R}^3$ in the image above), then W (the plane in the image) is a subset of V, and is called a subspace if:

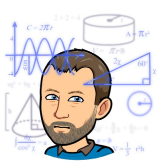
a) W is nonempty (it contains at least one vector),

b) Given $\vec{u}, \vec{v} \in W$, we have $\vec{u} + \vec{v} \in W$, (W is *closed under addition*)

c) Given $c \in \mathbb{R}$, we have $c\vec{u} \in W$. (W is *closed under scalar multiplication*)

And therefore $\vec{0} \in W$. Why?

Also, convince yourself that the x-axis and the y-axis (just the axes themselves, no other points), joined together as a subset of $\mathbb{R}^3$, *does not* constitute a subspace.



Does $c \Rightarrow a$?

Requirement a ensures that $W \neq \emptyset$.

Examples: Which of the following are subspaces?

$A := \{(x,y) \mid x \in [0,1]\} \subset \mathbb{R}^2$, …

$B := \{(x,y) \mid y = x\} \subset \mathbb{R}^2$, …

$C := \{(x,y,0)\} = \{(x,y,z) \mid z = 0\} \subset \mathbb{R}^3$, …

$D := \{\vec{0}\} \subset \mathbb{R}^n$, …

$$E := \left\{ \begin{bmatrix} x \\ y \\ 4 \end{bmatrix} : x,y \in \mathbb{R} \right\} \subset \mathbb{R}^3, \qquad \ldots$$

More Generally

Theorem: As a result of the subspace properties above, a subspace is a vector space in its own right - under the same operations of vector addition and scalar multiplication and the same 0 element.

Proof: The proof is immediate. For example, let us check **commutivity**.

Subspace elements $\vec{v}, \vec{w} \in W$ can be regarded as elements of V, in which case $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ because V is a vector space.

But the closure condition implies that sums also belongs to W, and so the commutivity axiom also holds for elements of W.

Establishing the validity of the other axioms is equally as easy. ■

Solution Subspace Thm: For $\mathbf{A}^{m\times n}$, the solution set of the homogeneous system $\mathbf{A}\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

Proof: Let W denote the solution set of the system. If $\vec{u}$ and $\vec{v}$ are vectors in W, then $\mathbf{A}\vec{u} = \mathbf{A}\vec{v} = \vec{0}$.

Hence: $\mathbf{A}(\vec{u} + \vec{v}) = \mathbf{A}\vec{u} + \mathbf{A}\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

Thus the sum $\vec{u} + \vec{v}$ is also in W, and hence W is closed under addition.

Next, if $c \in \mathbb{R}$, then $\mathbf{A}(c\vec{u}) = c(\mathbf{A}\vec{u}) = c\vec{0} = \vec{0}$.

Thus $c\vec{u}$ is in W if $\vec{u}$ is in W. Hence W is also closed under scalar multiplication.

Therefore, W is a subspace of $\mathbb{R}^n$. ■

Nonhomogeneous System Solutions Thm: The solution set of a nonhomogeneous system $\mathbf{A}\vec{x} = \vec{b}$ is **never** a subspace.

Proof: Let's do proof by contradiction. Let $\vec{u}$ be a solution in W, the set of solutions to $\mathbf{A}\vec{x} = \vec{b}$. And let us make the dubious assumption that W is a subspace.

Let $c = 0 \in \mathbb{R}$. By closure under scalar multiplication, $c\vec{u} = 0\vec{u} = \vec{0}$ is a solution.

Therefore, $\mathbf{A}\,\vec{0} = \vec{0} = \vec{b}$. (!?!)

But we assumed the system was nonhomogeneous: $\vec{b} \neq \vec{0}$.

So we have a contradiction, and our assumption that the set of solutions W was a subspace must have been incorrect. ■

Video Tutorial (visually rich and intuitive): https://youtu.be/fNk_zzaMoSs

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Exercises

Problem. Assume W is the set of all vectors in $\mathbb{R}^4$ such that $x_1 = 3x_3$ and $x_2 = 4x_4$. Apply the theorems in this section to determine whether or not W is a subspace of $\mathbb{R}^4$.

$W = \left\{\left(3c,\ 4d,\ c,\ d\right)\right\}$.

First, note that the subspace is nonempty since $(3, 4, 1, 1) \in W$, where $c, d = 1$.

We arbitrarily choose two vectors from W by arbitrarily choosing four constants $c_1, d_1, c_2, d_2 \in \mathbb{R}$, giving us $\left(3c_1,\ 4d_1,\ c_1,\ d_1\right)$ and $\left(3c_2,\ 4d_2,\ c_2,\ d_2\right)$. We then test them for closure under addition:

$$\left(3c_1,\ 4d_1,\ c_1,\ d_1\right) + \left(3c_2,\ 4d_2,\ c_2,\ d_2\right)$$

$$= \left(3c_1 + 3c_2,\ 4d_1 + 4d_2,\ c_1 + c_2,\ d_1 + d_2\right)$$

$$= \left(3(c_1 + c_2),\ 4(d_1 + d_2),\ c_1 + c_2,\ d_1 + d_2\right) \in W$$

This is because it has the prescribed format $\left\{\left(3c,\ 4d,\ c,\ d\right)\right\}$, where $c := c_1 + c_2$ and $d := d_1 + d_2$.

Now to test scalar multiplication:

$\alpha\left(3c_1,\ 4d_1,\ c_1,\ d_1\right) = \left(3\alpha c_1,\ 4\alpha d_1,\ \alpha c_1,\ \alpha d_1\right) \in W = \left\{\left(3c,\ 4d,\ c,\ d\right)\right\}$

where $c := \alpha c_1$ and $d := \alpha d_1$.

Therefore, W is nonempty, closed under addition, and scalar multiplication, and is a subspace of $\mathbb{R}^4$. ■

Problem. Reduce the given system to echelon form to find a single solution vector $\vec{u}$ such that the solution space is the set of all scaler multiples of $\vec{u}$.

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 0, \\ 2x_1 + 7x_2 + 5x_3 - \ x_4 &= 0, \end{aligned}$$

$$2x_1 + 7x_2 + 4x_3 - 4x_4 = 0.$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \\ 2 & 7 & 4 & -4 \end{bmatrix} \overset{\text{trust me}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus $x_4 = t$ is a parameter (a.k.a. free variable).

We solve for $x_1 = -6t,\ x_2 = 4t,$ and $x_3 = -3t.$ So,

$$\vec{x} = (x_1, x_2, x_3, x_4) = (-6t, 4t, -3t, t) = t\vec{u}, \text{ where } \vec{u} = (-6, 4, -3, 1).$$

Problem. For the following system of equations, find two solution vectors $\vec{u}$ and $\vec{v}$ such that the **solution space** is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$.

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \overset{\text{trust me}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_2 = -s - 3t,$ and $x_1 = -s - 5t.$ So …

$$\vec{x} = (x_1, x_2, x_3, x_4) = \left(-s - 5t,\ \ -s - 3t,\ s,\ t\right)$$

$$= (-s, -s, s, 0) + (-5t, -3t, 0, t) = s\vec{u} + t\vec{v}, \text{ where } \vec{u} = (-1, -1, 1, 0) \text{ and } \vec{v} = (-5, -3, 0, 1).$$

Problem. Let $\mathbf{A}$ be an $n \times n$ matrix, $\vec{b}$ be a nonzero vector, and $\vec{x}_0$ be a solution vector to the system $\mathbf{A}\vec{x} = \vec{b}$.

Show that $\vec{x_2}$ is another solution **iff** $\vec{x_2} - \vec{x}_0$ is a solution of the homogeneous system $\mathbf{A}\vec{y} = \vec{0}$.

We are given: $\mathbf{A}\vec{x}_0 = \vec{b}$.

Need to show that: $\mathbf{A}\vec{x_2} = \vec{b} \quad \Leftrightarrow \quad \mathbf{A}(\vec{x_2} - \vec{x}_0) = \vec{0}$.

Starting with the left assumption, and trying to show the thing on the right, we have:

$\mathbf{A}(\vec{x_2} - \vec{x}_0) = \mathbf{A}\vec{x_2} - \mathbf{A}\vec{x}_0 = \vec{b} - \vec{b} = 0.$ ✓

Going from right to left, we have: $\mathbf{A}(\vec{x_2} - \vec{x}_0) = \mathbf{A}\vec{x_2} - \mathbf{A}\vec{x}_0 = \mathbf{A}\vec{x_2} - \vec{b} = 0$, therefore $\mathbf{A}\vec{x_2} = \vec{b}$. **Q. E. D.**
