

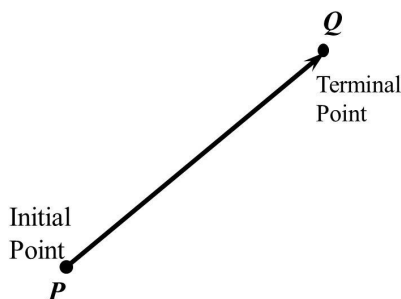
# Applied Linear Algebra

Textbook: *Applied Linear Algebra* by Olver and Shakiban

## 2.1 Real Vector Spaces

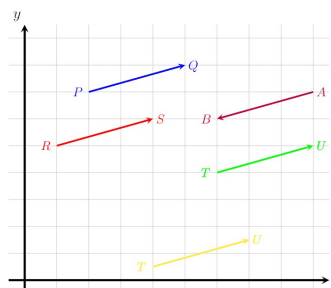
**Vector:** mathematical object consisting of **direction & magnitude**.

A vector, notated  $\vec{v}$ ,  $\mathbf{v}$ , or  $\overrightarrow{PQ}$ , is often drawn as an arrow starting at **initial point**  $P$  (tail), and ending at **terminal point**  $Q$  (tip, head).



**Magnitude:** length of arrow.

$\vec{u}, \vec{v}$  are equivalent ( $\vec{u} = \vec{v}$ ) if they share same magnitude & direction, even with different  $P$  (and therefore different  $Q$ ).



We sometimes write  $\vec{v} = \langle 2, 1 \rangle$  to mean  $(2, 1)$ . But  $\langle 2, 1 \rangle$  refers to any vector with same magnitude/direction located anywhere in  $\mathbb{R}^2$ .

**Zero Vector:**  $\mathbf{0}$  or  $\vec{0}$ , has length 0. Only vector with no (or any) direction.

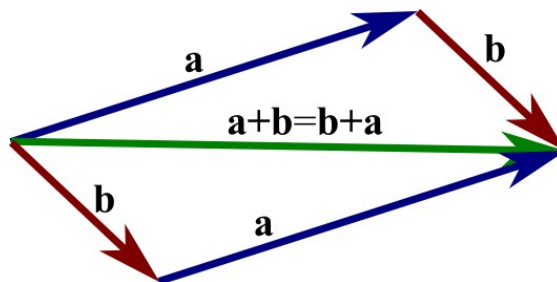
In 2D (similarly in 3D), if tail of  $\vec{a}$  is at the origin, then tip has coordinates  $(a_1, a_2)$ .

These are the **components of**  $\vec{a}$ . We write  $\vec{a} = \langle a_1, a_2 \rangle$ .

A *vector space* is a set  $V$  equipped with two operations:

**Vector Addition (Triangle Law):** adding any pair of vectors  $\vec{a}, \vec{b} \in V$  produces another vector  $\vec{a} + \vec{b} \in V$ ;

If  $\vec{a}, \vec{b}$  are positioned with tail of  $\vec{b}$  at tip of  $\vec{a}$ , then  $\vec{a} + \vec{b}$  is from tail of  $\vec{a}$  to tip of  $\vec{b}$ .

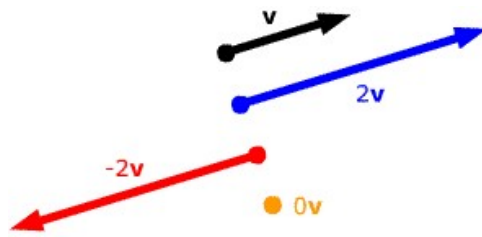


Geometric: Triangle & Parallelogram

**Scalar Multiplication:** multiplying a vector  $\vec{v} \in V$  by a scalar  $c \in \mathbb{R}$  produces a vector  $c\vec{v} \in V$ .

$$c\vec{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$$

Algebraic



Geometric

These are subject to the following axioms, valid for all  $\vec{u}, \vec{v}, \vec{w} \in V$  and all scalars  $c, d \in \mathbb{R}$ .

**Commutativity of Addition:**  $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

**Associativity of Addition:**  $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ .

**Additive Identity:** There is a zero element  $\vec{0} \in V$  satisfying  $\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$ .

**Additive inverse:** For each  $\vec{v} \in V$ , there is an element  $-\vec{v} \in V$  such that  $\vec{v} + (-\vec{v}) = \vec{0} = (-\vec{v}) + \vec{v}$ .

**Distributivity:**  $(c + d)\vec{v} = (c\vec{v}) + (d\vec{v})$ , and  $c(\vec{v} + \vec{w}) = (c\vec{v}) + (c\vec{w})$ .

**Associativity of Scalar Multiplication:**  $c(d\vec{v}) = (cd)\vec{v}$ .

**Unit for Scalar Multiplication:** the scalar  $1 \in \mathbb{R}$  satisfies  $1\vec{v} = \vec{v}$ .

Vector spaces are not limited to literal vectors! Example: polynomials of degree 3 or less.

$$P_3(x) := a_0 + a_1x + a_2x^2 + a_3x^3, \text{ for real } a_i.$$

A **vector space**, or more generally a **linear space** is a set of objects with a *reasonable* sense of addition and scalar multiplication.

**Polynomial spaces satisfy vector space properties:**

Given  $p_1, p_2, p_3$  in  $P_3$ .

$$\diamond p_1 + p_2 = p_2 + p_1$$

$$\diamond p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$$

$$\diamond p_1 + 0 = 0 + p_1 = p_1$$

$$\diamond p_1 + (-p_1) = (-p_1) + p_1 = 0$$

$$\diamond (c + k)p_1 = cp_1 + kp_1, \text{ for any real } c, k; \text{ and } k(p_1 + p_2) = kp_1 + kp_2, \text{ for any real } k$$

$$\diamond c(kp_1) = (ck)p_1$$

$$\diamond 1(p_1) = p_1$$

The following are elementary consequences of the vector space axioms:

$$\diamond 0\vec{v} = \vec{0},$$

$$\diamond (-1)\vec{v} = -\vec{v},$$

$$\diamond c\vec{0} = \vec{0},$$

$$\diamond \text{If } c\vec{v} = \vec{0}, \text{ then either } c = 0 \text{ or } \vec{v} = \vec{0}.$$

## Function Spaces (Linear Spaces of Functions)

Let  $F = \{ \text{all real valued functions on } \mathbb{R} \}$ ; includes all polynomials, trig functions, exponentials, etc... ..

Observe for  $\mathbf{f}, \mathbf{g}$  in  $F$ , we have:  $(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$  and  $(c\mathbf{f})(x) = c(\mathbf{f}(x))$ .

The properties of a linear space follow since these functions are real valued. ...

What is the neutral element?  $z(x)$  ...

## Linear Space of $m \times n$ Matrices

**Example:**  $2 \times 2$  matrices. Matrices like  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  with neutral element  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  and multiplicative identity element

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

...

---

# Infinite Dimensional Linear Spaces



$\mathbb{R}^\infty$ , space of vectors  $\vec{v} = (a_1, a_2, a_3, \dots)$ , where  $a_i$  in  $\mathbb{R}$ .

Polynomials:  $P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ , where  $a_i$  in  $\mathbb{R}$ , and ALL  $n$  in  $\{0, 1, 2, \dots\}$ .

Verify that they satisfy the axioms.