

Applied Linear Algebra

Textbook: *Applied Linear Algebra* by Olver and Shakiban

1.8 General Linear Systems

$$\begin{bmatrix} 0 & 7 & 3 & 4 & 1 \\ 0 & 0 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Zeros form a staircase

Definition: A matrix \mathbf{U} is said to be in **echelon form** provided it has the following two properties (staircase properties):

- Every row of \mathbf{U} that contains only zeros (if any) lies beneath every row that contains a nonzero element.
- In the other rows, the first nonzero element (**pivot**) lies strictly to the right of the first nonzero element in the preceding row (if there is a preceding row).

After reduction to echelon form, the variables corresponding to the columns with pivots, are referred to as **basic variables**.

The other variables are referred to as **free variables**.

$$\begin{bmatrix} 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So if our reduction resulted in the above echelon matrix, we would have 2 and 3 as pivots,

with x_2, x_4 as basic variables and x_1, x_3 as free variables.

Proposition: Every matrix can be reduced to row echelon form by a sequence of type 1 and/or type 2 row operations.

Definition: The *rank* of a matrix is the number of pivots. The rank of a matrix is the dimension of the vector space generated (or spanned) by its columns.

Proposition: $\mathbf{A}^{n \times n}$ is nonsingular **iff** its rank is equal to n .

Theorem: A system $\mathbf{A}\vec{x} = \vec{b}$ has either 0, 1, or ∞ solutions.

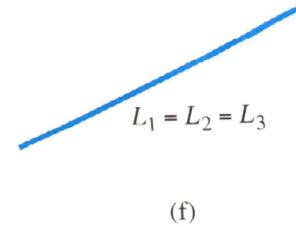
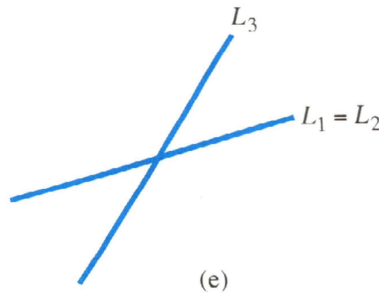
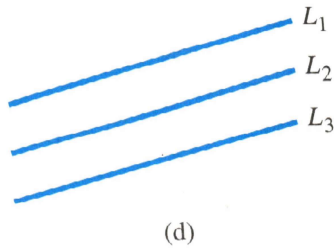
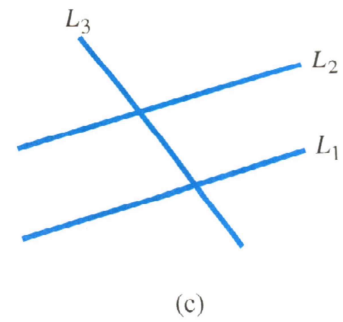
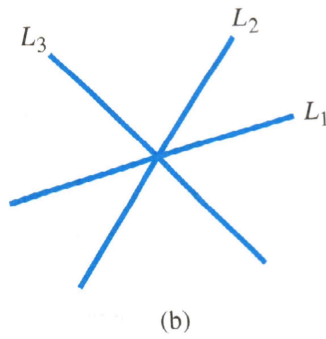
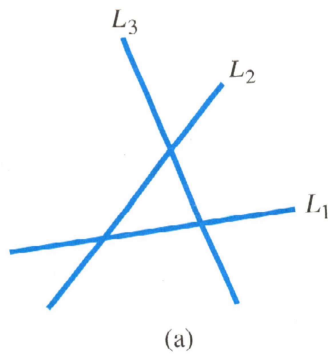
Example: $a_1x + b_1y = c_1$

The linear system: $a_2x + b_2y = c_2$

$$a_3x + b_3y = c_3$$

of three equations in two unknowns (x, y) represents three lines L_1 , L_2 , and L_3 in the xy -plane.

The figures below show possible configurations of these lines. Describe the solution set of each:



Homogeneous Systems

Definition: A system $A\vec{x} = \vec{b}$ is *homogeneous* if the b_1, b_2, \dots, b_n on the RHS are all zeros. Otherwise, it is *nonhomogeneous*.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 0 \\ a_{21} & a_{22} & a_{23} & | & 0 \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 1 \\ a_{21} & a_{22} & a_{23} & | & 0 \end{bmatrix}$$

homogeneous *nonhomogeneous*

Definition: A solution to $A\vec{x} = \vec{b}$ is called **trivial** if the solution consists of $x_1 = x_2 = \dots = x_n = 0$.

You should convince yourself that the trivial solution is always a solution to a homogeneous system.

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 0 \\ a_{21} & a_{22} & a_{23} & | & 0 \\ a_{31} & a_{32} & a_{33} & | & 0 \\ a_{41} & a_{42} & a_{43} & | & 0 \end{bmatrix} \Rightarrow \begin{matrix} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \\ a_{41}x + a_{42}y + a_{43}z = 0 \end{matrix} \Rightarrow \begin{matrix} a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 0 = 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 0 = 0 \\ a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 0 = 0 \\ a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 0 = 0 \end{matrix}$$

Observe this is true irrespective of what the a_{ij} are.

What if you knew that \mathbf{H} had at least one *nontrivial* solution?
 Could you conclude something about how many solutions it has?

What about a homogeneous linear system \mathbf{H} with more variables (columns) than equations (rows) how many solutions?

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 0 \\ a_{21} & a_{22} & a_{23} & | & 0 \end{bmatrix}$$

$$\rightarrow \quad \tilde{\mathbf{H}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & 0 \\ 0 & 1 & c_{23} & 0 \end{array} \right] ??$$

Theorem: A homogeneous system $\mathbf{A}\vec{x} = \vec{0}$ of m equations in n unknowns has a nontrivial solution $\vec{x} \neq \vec{0}$ **iff** the rank of \mathbf{A} is $r < n$.

If $m < n$, the system always has a nontrivial solution. If $m = n$, the system has a nontrivial solution **iff** \mathbf{A} is singular.