Applied Linear Algebra

Textbook: *Applied Linear Algebra* by Olver and Shakiban

1.6 Transposes and Symmetric Matrices

Transpose:

If $\mathbf{A} = \begin{bmatrix} \mathbf{c} & \mathbf{a} & \mathbf{t} \\ p & e & n \\ \mathbf{m} & \mathbf{o} & \mathbf{m} \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} \mathbf{c} & p & \mathbf{m} \\ \mathbf{a} & e & \mathbf{o} \\ \mathbf{t} & n & \mathbf{m} \end{bmatrix}$.

matrix_transpose_animation.gif

See animation in class

In particular: $\mathbf{B} = \mathbf{A}^T$ means that $b_{ij} = a_{ji}$.

Most vectors are column vectors, but to conserve space in text, these are often written as $(a_1, \ldots, a_n)$ or $\begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}^T$.

Observe that the transpose of a lower triangular matrix is upper triangular.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \overset{T}{\to} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Observe that a scalar (1×1 matrix) is its own transpose.

Special case: Consider the row vector $\vec{v}^T = [v_1 \ldots v_n]$ and a column vector $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ of the same size.

Multiplying these: $\vec{v}^T \vec{w} \overset{?}{=} \left(\vec{v}^T \vec{w}\right)^T = \vec{w}^T \vec{v}$.

The first equal sign is justified because their product is a scalar, and a scalar is its own transpose.

$$38 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 38$$

Transpose Properties:

- ♦ $(\mathbf{A}^T)^T = \mathbf{A}$
- ♦ $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
- ♦ $(c\mathbf{A})^T = c\mathbf{A}^T$
- ♦ $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ (recall: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$)

Lemma: If $\mathbf{A}$ is a nonsingular matrix, so is $\mathbf{A}^T$, and its inverse is denoted by: $\mathbf{A}^{-T} := (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

Proof: Let $\mathbf{X} = (\mathbf{A}^{-1})^T$. Must show $\mathbf{X}\mathbf{A}^T = \mathbf{I} = \mathbf{A}^T\mathbf{X}$.

We have $\mathbf{X}\mathbf{A}^T = (\mathbf{A}^{-1})^T\mathbf{A}^T = (\mathbf{A}\mathbf{A}^{-1})^T$ (since $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$)

So, $\mathbf{X}\mathbf{A}^T = \mathbf{I}^T = \mathbf{I}$.

The proof that $\mathbf{A}^T\mathbf{X} = \mathbf{I}$ is similar, and so $(\mathbf{A}^{-1})^T$ is the inverse of $\mathbf{A}^T$.

Note that we have also shown with the same calculations that $\mathbf{X} = (\mathbf{A}^T)^{-1}$. ■

Factorization of Symmetric Matrices

Definition: A matrix is called symmetric if it equals its own transpose: $\mathbf{A} = \mathbf{A}^T$.

Symmetric matrix

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 8 & 4 \\ 6 & 4 & 5 \end{bmatrix}$$

Theorem: A symmetric matrix $\mathbf{A}$ is regular **iff** it can be factored as $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, where $\mathbf{L}$ is a lower unitriangular matrix and $\mathbf{D}$ is a diagonal matrix with nonzero diagonal entries.

Proof: $\Leftarrow$ This is trivial since $\mathbf{L}^T$ is upper triangular, and so by previous thm ($\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{V}$) $\mathbf{A}$ is regular.

$\Rightarrow$ We already know that we can factor $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{V}$ (since $\mathbf{A}$ regular) $(*)$

We take the transpose of this equation: $\mathbf{A}^T = (\mathbf{L}\mathbf{D}\mathbf{V})^T = \mathbf{V}^T\mathbf{D}^T\mathbf{L}^T = \mathbf{V}^T\mathbf{D}\mathbf{L}^T$, $(*\,*)$
(since diagonal matrices are automatically symmetric).

Observe that $\mathbf{V}^T$ is lower unitriangular, and $\mathbf{L}^T$ is upper unitriangular.
Therefore, $(*\,*)$ is the $\mathbf{LDV}$ factorization of $\mathbf{A}^T$.

In particular, if $\mathbf{A}$ is symmetric, then: $\mathbf{L}\mathbf{D}\mathbf{V} = \mathbf{A} = \mathbf{A}^T = \mathbf{V}^T\mathbf{D}\mathbf{L}^T$.

Uniqueness of the $\mathbf{LDV}$ factorization implies that $\mathbf{L} = \mathbf{V}^T$ and $\mathbf{V} = \mathbf{L}^T$.

Replacing $\mathbf{V}$ by $\mathbf{L}^T$ in $(*)$ establishes the factorization $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$. ■

True or false: If $\mathbf{A}$ is symmetric, then $\mathbf{A}^2$ is symmetric.

Recall: $\mathbf{AB} = [c_{ij}]$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

So, if $\mathbf{B} = \mathbf{A}$, then $\mathbf{A}^2 = [c_{ij}]$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$ and

$c_{ji} = \sum_{k=1}^{n} a_{jk}a_{ki}$ (but we get to use the symmetry of $\mathbf{A}$)

$= \sum_{k=1}^{n} a_{ik}a_{kj} = c_{ij}$. (symmetry of $\mathbf{A}$ implies $a_{ik} = a_{ki}$ and $a_{kj} = a_{jk}$)

And so $\mathbf{A}^2$ is symmetric. ■

True or false: If $\mathbf{A}$ is a nonsingular symmetric matrix, then $\mathbf{A}^{-1}$ is also symmetric.

Since $\mathbf{A}$ is symmetric, then $\mathbf{A}^T = \mathbf{A}$. Must show $(\mathbf{A}^{-1})^T = \mathbf{A}^{-1}$.

Observe that $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} = (\mathbf{A})^{-1} = \mathbf{A}^{-1}$. ■