MATH 2243: Linear Algebra & Differential Equations

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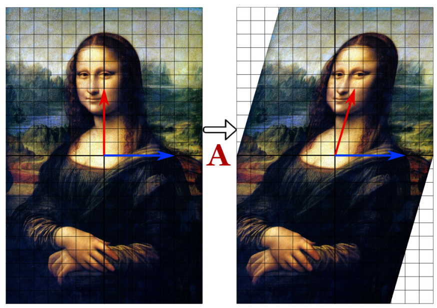
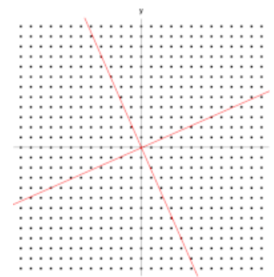
6.1: Eigenvalues

For an $n \times n$ matrix $\mathbf{A}$, if we have: $\mathbf{A}\vec{v} = \lambda\vec{v}$, then the scalar λ is an **eigenvalue**, and $\vec{v}$ is an **eigenvector**, where $\vec{v}$ is a nonzero vector, $\lambda \in \mathbb{C}$, and $\mathbf{I}$ is the identity matrix (or equavalently $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$).



vector (red), eigenvector (blue) under $\mathbf{A}$



(see animated during class)

Motivation: Recall how points in the $\mathbb{R}^n$ vector space are characterized as linear combinations of the unit vectors u_x along the axes; $(1,-3,5) = 1(1,0,0) - 3(0,1,0) + 5(0,0,1) = 1u_x - 3u_y + 5u_z$.

Analogously, solutions in the solution space to a system of DEQs $\vec{x}' = \mathbf{A}\vec{x}$ can be characterized as linear combinations of $e^{\lambda t}\vec{v}_\lambda$, where $\lambda, \vec{v}_\lambda$ are the eigenvalues, eigenvectors of $\mathbf{A}$ (we will learn about this in chapter 7).

Applications: Modeling migration patterns, predator-prey relationships, fluid dynamic, and many more.

Calculation

Given $\mathbf{A}$, how do we find its eigenvalues and eigenvectors?

We want $\lambda, \vec{v}$ such that: $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$.

Observe that this is a homogeneous system of n equations, where our n unknowns are the components of $\vec{v}$.

Recall that such a system has a nontrivial solution ($\vec{v} \neq 0$) when $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

So by imposing this restriction, we can identify the various λ.

Characteristic Equation: $|\mathbf{A} - \lambda\mathbf{I}| =$

$$= \left| \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{(n-1)n} \\ a_{n1} & \dots & a_{n(n-1)} & a_{nn} - \lambda \end{vmatrix} = 0.$$

Finding eigenvalues and eigenvectors of $\mathbf{A}$:

- ♦ Solve $|\mathbf{A} - \lambda\mathbf{I}| = c_1\lambda^n + c_2\lambda^{n-1} \ldots + c_0 = 0,$ for all λ_k (should be n of them, including multiplicity)

  Example: Prob. 21 below

- ♦ Then, for each λ_k, solve $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}$ to find the eigenvector(s) $\vec{v}$ for λ_k.

$$(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \begin{bmatrix} a_{11} - \lambda_k & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} - \lambda_k \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}.$$

  Example: Prob. 21 below

If the components of $\mathbf{A}$ are real, then any complex eigenvalues will occur in conjugate pairs (i.e., $\lambda_\pm = a \pm bi$).

  Example: Prob. 30 below

Eigenspaces:

Each eigenvalue λ_k associated with $\mathbf{A}$ will produce a set of (one or more) linearly independent eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m\}$.

For each λ_k, the associated eigenvectors form a basis for a subspace, $span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m\} = \mathbb{R}^m \subseteq \mathbb{R}^n$.

This subspace is called an **eigenspace**, and is FULL of eigenvectors which are linear combinations of the discovered basis.

The eigenspace of each λ_k serves as the solution space to $(\mathbf{A} - \lambda_k I)\vec{v} = \vec{0}$.

Video Tutorial (visually rich and intuitive): https://youtu.be/PFDu9oVAE-g

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Exercises 💪

Problem: # 21   Find the (real) eigenvalues, the associated eigenvectors, and a basis for each eigenspace for the matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)((4-\lambda)(-1-\lambda)+6) \qquad \text{(pro tip....)}$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2) \quad = \quad -(\lambda-1)(\lambda-2)^2.$$

Characteristic Polynomial: $p(\lambda) = -(\lambda-1)(\lambda-2)^2 = 0.$

Eigenvalues: $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = 2.$ Now what?

For each λ_k, solve $(\mathbf{A} - \lambda_k \mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_1 = 1$:
$$\begin{bmatrix} 4-1 & -3 & 1 \\ 2 & -1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1+(-1)R_2}{\Rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_2+(-1)R_1}{\Rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad z = 0,\ y = b,\ x = y = b.$$

$$\Rightarrow \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ when } b = 1.$$

The eigenspace of $\lambda_1 = 1$ is 1-dimensional.
Basis for λ_1 eigenspace: $\{\vec{v}_1\}$.

With $\lambda_{2,3} = 2$: $\quad \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 4-2 & -3 & 1 \\ 2 & -1-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix}$

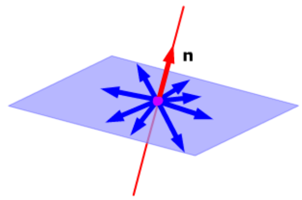
$$= \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, \quad z = c, \quad y = b, \quad x = \tfrac{3}{2}y - \tfrac{1}{2}z \ = \ \tfrac{3}{2}b - \tfrac{1}{2}c.$$

$$\Rightarrow \begin{bmatrix} \frac{3}{2}b - \frac{1}{2}c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \text{ and } \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \text{ when } b, c = 2.$$

The eigenspace of $\lambda_{2,3} = 2$ is two-dimensional.
Basis for $\lambda_{2,3}$ eigenspace: $\{\vec{v}_2, \vec{v}_3\}$.



Problem: **#30** Find the complex-conjugate eigenvalues and corresponding eigenvectors of the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix}.$$

Characteristic polynomial: $p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 0-\lambda & -12 \\ 12 & 0-\lambda \end{vmatrix}$

$$= \lambda^2 + 144 = 0.$$

Eigenvalues: $\lambda_1 = -12i,\ \lambda_2 = +12i.$

For each λ_k, solve $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_1 = -12i$: $\begin{bmatrix} 0-\lambda_1 & -12 \\ 12 & 0-\lambda_1 \end{bmatrix} = \begin{bmatrix} 12i & -12 \\ 12 & 12i \end{bmatrix}$

$$\overset{\frac{1}{12}R_{1,2}}{\Rightarrow} \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Rightarrow} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow\ y = b \text{ and } x = -ib.$$

So, $\vec{v}_1 = \begin{bmatrix} -ib \\ b \end{bmatrix} = b\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, when $b = 1$.

Similarly…

With $\lambda_2 = +12i$: $\left.\begin{array}{r} -12ia - 12b = 0 \\ 12a - 12ib = 0 \end{array}\right\}$ $\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

(leave it to you as an exercise)

Note that $\vec{v}_1$ and $\vec{v}_2$ are conjugate to each other.

Problem: #35 **a) Suppose that A is a square matrix.**

Use the characteristic equation to show that A and $\mathbf{A}^T$ have the same eigenvalues.

Note first that $(\mathbf{A} - \lambda\mathbf{I})^T = (\mathbf{A}^T - \lambda\mathbf{I}^T) = (\mathbf{A}^T - \lambda\mathbf{I})$, because $\mathbf{I}^T = \mathbf{I}$.

Since we learned earlier that the determinant of a square matrix equals the determinant of its transpose,

it follows that $|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{A}^T - \lambda\mathbf{I}|$.

This means the matrices **A** and $\mathbf{A}^T$ have the same characteristic polynomial. Therefore they have the same eigenvalues.

b) Give an example of a 2 × 2 matrix A such that A and $\mathbf{A}^T$ do not have the same eigenvectors.

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ with characteristic equation $(\lambda - 1)^2 = 0$ and the single eigenvalue $\lambda = 1$.

Then $\mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and it follows that the only associated eigenvector is a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The transpose $\mathbf{A}^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the same characteristic equation and eigenvalue,

but $\mathbf{A}^T - \mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so its only eigenvector is a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus **A** and $\mathbf{A}^T$ have the same eigenvalue but different eigenvectors.