MATH 2243: Linear Algebra & Differential Equations

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5.3: Homogeneous DEQs with Constant Coefficients

$y^{(n)} + d_1 y^{(n-1)} + \ldots + d_{n-1} y' + d_n y = 0 \qquad (*)$

For an nth-order homogeneous DEQ, we know that if we can find n linearly independent solutions, we can form a general solution.

From the observation that $(e^{rt})^{(n)} = r^n e^{rt}$, we suspect solutions like $y = e^{rt}$ for some $r \in \mathbb{R}$.

Substituting e^{rt} into $(*)$, we get: $(e^{rt})^{(n)} + d_1 (e^{rt})^{(n-1)} + \ldots + d_{n-1}(e^{rt})' + d_n(e^{rt})$

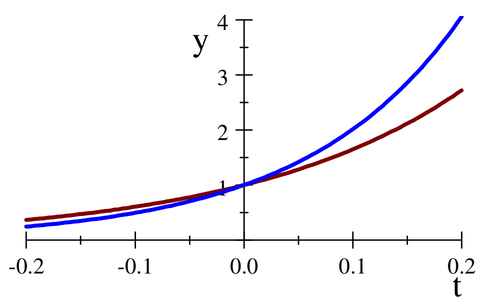
$= r^n e^{rt} + d_1 r^{n-1} e^{rt} + \ldots + d_{n-1} r e^{rt} + d_n e^{rt}$

$= e^{rt}(r^n + d_1 r^{n-1} + \ldots + d_{n-1} r + d_n) = 0.$

However, e^{rt} is never zero, so we need only find r satisfying the **characteristic equation**:

$r^n + d_1 r^{n-1} + \ldots + d_{n-1} r + d_n = 0 \qquad (**).$

Observe that if $r_1 \neq r_2$, then $e^{r_1 t} \neq k e^{r_2 t}$, for any $k \in \mathbb{R}$. Therefore, $e^{r_1 t}, e^{r_2 t}$ are independent.



e^{5t} and e^{7t}

Proof: If $e^{r_1 t} = k e^{r_2 t}$, then

$r_1 t = \ln|k e^{r_2 t}| = \ln|k| + r_2 t$

$\Rightarrow \quad t(r_1 - r_2) = C$

$\Rightarrow \quad t = \frac{C}{r_1 - r_2}$, but the left-hand side is not constant, while the right-hand side is constant. $\blacksquare$

Distinct Real Roots Theorem: With real distinct roots $r_1, r_2, \ldots, r_n$ of the characteristic equation, a general solution is:

$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$. And, $\left\{ e^{r_1 x},\ e^{r_2 x}, \ldots,\ e^{r_n x} \right\}$ is a basis for the solution space to $(*)$.

Differential Operator L acting on $f(x)$: (?!?!?!)

We are used to seeing functions f, g (as operators) that operate on *variables* (x, y), by raising them to some power, adding them in linear combinations, etc. (e.g., $f(x) = 7x^2 + 2x + 7$). But we can also define operators L that operate on *functions* by taking some derivatives, adding them in linear combinations, etc (e.g., $L(f(x)) = 7f'' + 2f + 7$).

Identity operator I: $I(f) = f$.

Derivative operator D^i: $D^2(f) = f''$.

So if $a_2y'' + a_1y' + a_0y = 0$, we can rewrite as:

$$(a_2D^2 + a_1D + a_0I)y = L(y) = 0, \text{ where } L := (a_2D^2 + a_1D + a_0I) \text{ is a } \textit{differential operator}.$$

Having defined a differential operator L, we can write DEQs more succinctly as $Ly = 0$.

Repeated Roots

Recall: $y^{(n)} + d_1y^{(n-1)} + \ldots + d_{n-1}y' + d_ny = 0 \qquad (*)$

$r^n + d_1r^{n-1} + \ldots + d_{n-1}r + d_n = 0 \qquad (**)$

Repeated Roots (of the characteristic equation): If a root r has multiplicity k (as in $(x-r)^k$), then the corresponding part of the general solution has the form: $(c_1 + c_2x + \ldots + c_kx^{k-1})e^{rx}$.

Proof: Fundamental theorem of algebra guarantees $r_1, \ldots, r_n$ roots to $(**)$.

If we assume r_1 is our repeated root, then $(**)$ can be rewritten as:

$$(r-r_1)^k(r-r_2)^{n_2} \bullet \ldots \bullet (r-r_j)^{n_j} = 0, \text{ where } k + n_2 + \ldots + n_j = n.$$

We need only focus on $(r-r_1)^k$.

Corresponding operator L in $(*)$ is $(D-r_1)^k$,

and the requirement on y for these roots is that $(D-r_1)^ky = 0$.

We know e^{r_1x} is a solution, and we suspect other linearly independent

versions of this solution like: $y = u(x)e^{r_1x}$.

Substituting this into the requirement, we have:

$$(D-r_1)^k[ue^{r_1x}]$$

$$= (D-r_1)^{k-1}(D-r_1)[ue^{r_1x}]$$

$$= (D-r_1)^{k-1}(D(ue^{r_1x}) - r_1ue^{r_1x})$$

$$= (D - r_1)^{k-1}(e^{r_1x}Du + uD(e^{r_1x}) - r_1ue^{r_1x}) \qquad \text{(product rule)}$$

$$= (D - r_1)^{k-1}(e^{r_1x}Du + r_1ue^{r_1x} - r_1ue^{r_1x})$$

$$= (D - r_1)^{k-1}e^{r_1x}Du$$

$$\vdots$$

$$= e^{r_1x}D^ku = 0.$$

But we know $e^{r_1x} \neq 0$, so we need $D^ku = 0$.

If we assume $u(x)$ is a polynomial, the largest order polynomial that would satisfy this is:

$$c_kx^{k-1} + c_3x^{k-2} + \ldots + c_2x + c_1. \qquad (***)$$

But any smaller order polynomial would also satisfy the requirement.

Indeed, a basis of functions which would span this function space would be: $\{1, x, \ldots, x^{k-1}\}$.

Putting these in a linear combination gives you $(***)$.

So the solutions $y = u(x)e^{r_1x}$ would include $e^{r_1x}, xe^{r_1x}, \ldots, x^{k-2}e^{r_1x}, x^{k-1}e^{r_1x}$.

Note that there are k of them, and they are linearly independent.

$$(e^{r_1x}x^{k_1} = Ce^{r_1x}x^{k_2} \quad \Rightarrow \quad x^{k_1-k_2} = C)$$

Therefore, we can add them in a linear combination to find the full set of solutions for r_1 :

$$c_kx^{k-1}e^{r_1x} + c_{k-1}x^{k-2}e^{r_1x} + \ldots + c_2xe^{r_1x} + c_1e^{r_1x}. \qquad \blacksquare$$

Complex Roots

A happy consequence of restricting ourselves to real coefficients in our DEQ is that the characteristic equation also has real coefficients, and thus any complex roots of this characteristic equation come in *conjugate pairs*. In other words, if you find a root $3 + 2i$, you're guaranteed to also have the root $3 - 2i$.

But how do we formulate solutions to our DEQ when the roots are complex?

Complex Numbers ($z \in \mathbb{C}$):

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Proof: Let $f(\theta) = e^{-i\theta}(\cos\theta + i\sin\theta)$.

Differentiating: $f'(\theta) = -ie^{-i\theta}(\cos\theta + i\sin\theta) + e^{-i\theta}(-\sin\theta + i\cos\theta) = 0.$

Since the derivative $f'(\theta)$ is 0, this means that $f(\theta)$ is a constant.

Also note that $f(0) = 1$. Therefore, $f(\theta) = 1$ for all θ.

Therefore, $1 = e^{-i\theta}(\cos\theta + i\sin\theta)$, or equivalently $e^{i\theta} = \cos\theta + i\sin\theta$. ■

So, if $z = 7 + 3i$, then $e^{zt} = e^{7t+3it} = e^{7t}e^{3it}$

$= e^{7t}(\cos 3t + i\sin 3t) = e^{7t}\cos 3t + ie^{7t}\sin 3t$.

$e^{7t}\cos 3t$ is the **real part** of the number e^{zt}, and

$e^{7t}\sin 3t$ is called the **imaginary part** (even though $e^{7t}\sin 3t$ is a real number!!).

Complex-Valued Functions:

For $F(t) = f(t) + ig(t)$ (note that the values taken on by the function are complex)

we can take the derivative as: $F'(t) = f'(t) + ig'(t)$.

In order for $F(t)$ to be the solution of a homogeneous DEQ ($L[F(t)] = 0$),

it is necessary that both the real part and the imaginary part of the function

are **also** each solutions. In other words: $L[f(t)] = 0$ AND $L[g(t)] = 0$.

Since (in this course) we are interested in only real valued functions, if a characteristic equation has complex conjugate roots ($7 \pm 3i$), the corresponding part of the general solution would take the form: $e^{7x}(c_1\cos 3x + c_2\sin 3x)$. Note how we have removed the i from sin, and instead given both solutions arbitrary real constants. So we still get two linearly independent solutions, but now they are both real instead of having complex solutions.

Repeated Complex Roots (**of the characteristic equation**):

If the roots ($5 \pm 7i$) have multiplicity 2, the relevant part of the general solution has the form:

$$e^{5x}(c_1\cos 7x + c_2\sin 7x) + xe^{5x}(c_3\cos 7x + c_4\sin 7x).$$

Or equivalently:it's

$$(c_1 + c_3x)e^{5x}\cos 7x + (c_2 + c_4x)e^{5x}\sin 7x.$$

PRO TIP

When finding the general solution to an nth order DEQ, you should find n arbitrary constants $(c_1, \ldots, c_n)$.

If you don't, you've made an error. This is a good way to check your work!

Exercises

Problem: #34 One solution to: $3y^{(3)} - 2y'' + 12y' - 8y = 0$, is $y = e^{\frac{2x}{3}}$. Find the general solution.

Characteristic equation: $3r^3 - 2r^2 + 12r - 8$.

Knowing that $y = e^{\frac{2x}{3}}$ is one solution, we know one factor:

$\left(r - \frac{2}{3}\right)$ or $(3r - 2)$.

I'm going to use polynomial division! $\frac{3r^3 - 2r^2 + 12r - 8}{3r - 2}$ or $3r^3 - 2r^2 + 12r - 8 = (3r - 2) \cdot (?????)$

$3r^3 - 2r^2 + 12r - 8 = (3r - 2)r^2 + (12r - 8)$

Further dividing the remainder: $12r - 8 = 4(3r - 2) + 0$.

So, we get: $3r^3 - 2r^2 + 12r - 8 \;=\; r^2(3r - 2) + 4(3r - 2) \;=\; (3r - 2) \cdot (r^2 + 4)$

And since $(r^2 + 4) = (r - 2i)(r + 2i)$, our two remaining solutions are obtained from:

e^{-2ix} and e^{2ix}, or ...

$\cos 2x - i \sin 2x, \qquad \cos 2x + i \sin 2x.$

Linearly independent solutions from above (real and imaginary parts) are: $\cos 2x$ and $\pm \sin 2x$.

Hence our general solution is $y(x) = c_1 e^{\frac{2x}{3}} + c_2 \cos 2x + c_3 \sin 2x$.

Problem #26 Solve the initial value problem: $y^{(3)} + 10y^{(2)} + 25y' = 0$, with initial conditions: $y(0) = 3$, $y'(0) = 4$, and $y''(0) = 5$.

$r^3 + 10r^2 + 25r = 0,$

$r(r^2 + 10r + 25) = 0,$

$r(r + 5)^2 = 0.$

$r = 0$, and $r = -5$ with multiplicity 2.

So: $y = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}. \qquad (*)$

Plugging in the initial condition $y(0) = 3$:

$3 = c_1 + c_2. \qquad (**)$

$$c_1 = 3 - c_2 \quad \Rightarrow \quad y = 3 - c_2 + c_2 e^{-5x} + c_3 x e^{-5x}.$$

$$y' = -5c_2 e^{-5x} - 5c_3 x e^{-5x} + c_3 e^{-5x}.$$

Plugging in the initial condition $y'(0) = 4$:

$4 = -5c_2 + c_3 \qquad\qquad (* * *)$

$$c_3 = 4 + 5c_2 \quad \Rightarrow \quad y' = -5c_2 e^{-5x} - (20 + 25c_2) x e^{-5x} + (4 + 5c_2) e^{-5x} = e^{-5x}(4 - 5(5c_2 + 4)x).$$

$$y'' = -5e^{-5x}(4 - 5(5c_2 + 4)x) + e^{-5x}(0 - 5(5c_2 + 4))$$

Plugging in the initial condition $y''(0) = 5$:

$5 = -5(4 - 0) - 5(5c_2 + 4) \quad \Rightarrow \quad 5 = -40 - 25c_2 \quad \Rightarrow \quad c_2 = -\frac{45}{25} = -\frac{9}{5}.$

Plugging this into $(* * *)$, we get $c_3 = 4 + 5(-\frac{9}{5}) = -5.$

And plugging into $(* *)$, we get $3 = c_1 - \frac{9}{5} \quad \Rightarrow \quad c_1 = \frac{24}{5}.$

And plugging all of these into our general solution $(*)$, we have the particular solution: $y_p = \frac{24}{5} + -\frac{9}{5} e^{-5x} - 5xe^{-5x}.$

Problem: #19 **Find the general solution of**: $y^{(3)} + y'' - y' - y = 0.$

$r^3 + r^2 - r - 1 = 0.$

How to solve? Educated guesses for roots to polynomials of this form, having a constant term c are $\pm$ the factors of c. For instance, if you had a polynomial $r^3 + r^2 - r - 6 = 0$, some good educated guesses would be $r \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$ (the factors of 6).

For our problem, observe that $r = 1$ is a root (observe $r = -1$ works too). So dividing:

$$r^3 + r^2 - r - 1 = (r - 1)r^2 + (2r^2 - r - 1)$$

Further dividing the remainder: $2r^2 - r - 1 = (r - 1)2r + (r - 1) = (r - 1)(2r + 1)$

Therefore, $r^3 + r^2 - r - 1 = (r - 1)(r^2 + 2r + 1) = (r - 1)(r + 1)^2.$

So, $y = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}.$

Problem #31 **Find a general solution of** $y^{(3)} + 3y'' + 4y' - 8y = 0$.

$r^3 + 3r^2 + 4r - 8 = 0$. Educated guesses: $\{\pm1, \pm2, \pm4, \pm8\}$.

$r = 1$ works.

So: $r^3 + 3r^2 + 4r - 8 = (r-1)r^2 + (4r^2 + 4r - 8)$

Further dividing the remainder: $4r^2 + 4r - 8 = (r-1)4r + (8r - 8) = (r-1)4r + 8(r-1) = (r-1)(4r+8)$

Therefore, $r^3 + 3r^2 + 4r - 8 = (r-1)r^2 + (r-1)(4r+8) = (r-1)(r^2 + 4r + 8)$

$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 8}}{2} = -2 \pm 2i$. So, $r \in \{1, -2 \pm 2i\}$.

$e^{(-2+4i)x} = e^{-2x}(\cos 4x + i \sin 4x)$.

Linearly independent solutions from above (real and imaginary parts) are: $e^{-2x}\cos 4x$ and $e^{-2x}\sin 4x$.

So putting our three solutions together, we have $y = c_1 e^x + c_2 e^{-2x}\cos 4x + c_3 e^{-2x}\sin 4x$

$= c_1 e^x + (c_2 \cos 4x + c_3 \sin 4x)e^{-2x}$

Problem: **#42** Find a **linear homogeneous constant coefficient** DEQ with the given general solution.

$$y(x) = (A + Bx + Cx^2)\cos 2x + (D + Ex + Fx^2)\sin 2x$$

Must've come from something like: $\cos 2x + i \sin 2x = e^{2ix}$ or $\cos 2x - i \sin 2x = e^{-2ix}$

$(r - 2i), (r + 2i)$ must be factors.

$(r - 2i)(r + 2i) = (r^2 + 4)$, What next?

$(r^2 + 4)^3 = r^6 + 12r^4 + 48r^2 + 64$, Are we done?

So the DEQ is: $y^{(6)} + 12y^{(4)} + 48y'' + 64y = 0$.

Problem: **#44** **DEQs with complex coefficients**.

Use the quadratic formula to solve the following equations.

Note in each case that the roots are not complex conjugates.

a) $x^2 + ix + 2 = 0$

$x = \frac{-i \pm \sqrt{-1-8}}{2} = -\frac{1}{2}i \pm \frac{3}{2}i = i$ or $-2i$.

b) $x^2 - 2ix + 3 = 0$

$$x = \frac{2i \pm \sqrt{-4-12}}{2} = i \pm 2i = 3i \text{ or } i.$$

Note: **Section 5.4 is a doozy**, **so you may wish to read ahead**.