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Big idea: Knowing the relationship between bases, dimensionality, and independence of vectors gives us information about solution sets of homogeneous linear systems, and vice versa.

4.4: Bases and Dimensions for Vector Spaces

Solution sets of homogeneous systems can be succinctly represented as a set of vectors, whose linear combinations give all possible solutions. We call this set a **basis**.

Let vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ exist in the vector space V.

Basis: S is called a basis for V if the vectors in S are linearly independent, and span V.

Standard Basis for $\mathbb{R}^n$: $\vec{e}_1 = (1,0,0,\ldots,0),\ \vec{e}_2 = (0,1,0,0,\ldots,0),\ \ldots,\ \vec{e}_n = (0,0,\ldots,1)$.

Sufficient Vectors for Basis Theorem: Any set of n linearly independent vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Proof: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be n linearly independent vectors in $\mathbb{R}^n$.

From previous section, we know that any set of more than n vectors in $\mathbb{R}^n$ is linearly dependent.

Therefore, given any vector $\vec{w}$ in $\mathbb{R}^n$, there exist scalars $c, c_1, c_2, \ldots, c_n$ not all zero such that:

$$c\vec{w} + c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}.$$

If c were zero, then this equation would imply that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent.

Therefore, $c \neq 0$. So, this equation can be solved for $\vec{w}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Thus, the linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ also span $\mathbb{R}^n$ and constitute a basis for $\mathbb{R}^n$. ■

Vector Space Dimensions

The **dimension of a vector space** is the number of vectors in its basis.

Bases as Maximal Linearly Independent Sets Theorem: If you have a **basis** S (for n-dimensional V) consisting of n vectors, then any set S' having more than n vectors is linearly dependent.

Dimension of a Vector Space Theorem: Any two bases for a vector space have the same number of vectors.

Proof: Let $S := \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $T := \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be two different bases for the same vector space V.

Because S is a basis and T is linearly independent, the previous theorem implies $m \leq n$.

Next, since T is a basis and S is linearly independent: $n \leq m$.

So: $m = n$. ■

Infinite Dimensional Vector Space P

Polynomials of the form: $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$.

Example vectors in P : $\left\{0,\ x,\ -7,\ 2+x^4,\ 7+x-x^{13}\right\}$.

Easily shown that P is a vector space.

Note that one basis for polynomials is $\{p_1, p_2, p_3, \ldots\} = \{1, x, x^2, \ldots\}$,

and all other bases have the same number of elements (Dimension of a Vector Space Theorem).

The dimension cannot be finite.

Proof: Proof by contradiction. Assume $\dim(P) = n < \infty$. So there are n vectors $B = \{p_1, p_2, \ldots, p_n\}$ in the basis.

Observe that the degree of any linear combination of the p_i is at most the maximum of their degrees.

Assume this maximum is m.

Observe that the polynomial x^{m+1} is in P, and can't be formed by a linear combination of the p_i.

So B can't be the basis for P, and our assumption that P is finite dimensional was incorrect. ■

A nonzero vector space that has no finite basis is called **infinite dimensional**.

Relationship between Spanning/Independence/Bases

Let V be an n-dimensional vector space and let S be a subset of V. Then:

- ♦ If S is linearly independent and consists of n vectors, then S is a basis for V. (we have enough vectors)

- ♦ If S spans V and consists of n vectors, then S is a basis for V. (we don't have too many vectors)

- ♦ If S is linearly independent, then S is contained in a basis for V. (we may need more vectors)

- ♦ If S spans V, then S contains a basis for V. (we may have too many vectors)

Finding the Solution Space Basis

Given the homogeneous linear equation $\mathbf{A}\vec{x} = \vec{0}$, to find the solution space W we:

- ♦ Reduce the coefficient matrix $\mathbf{A}$ to echelon form.

- ♦ Identify the r leading variables $(x_1, \ldots, x_r)$ and
  the $k = n - r$ free variables $(x_{r+1}, \ldots, x_n)$. If $k = 0$, then $W = \left\{\vec{0}\right\}$.

- ♦ Set the free variables equal to parameters $t_1, t_2, \ldots, t_k$.

- ♦ Solve by back substitution for the leading variables in terms of these
  parameters.

- ♦ For each $1 \leq j \leq k$, let $\vec{v}_j$ be the solution vector obtained by
  setting $t_j = 1$, and the other parameters equal to zero.

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for the solution space W.

Video Tutorial (visually rich and intuitive): https://youtu.be/kYB8IZa5AuE

Exercises

Problem: **# 7** Determine whether or not the given vectors in $\mathbb{R}^4$ form a basis for $\mathbb{R}^4$.

$\vec{v}_1 = (2,0,0,0)$, $\vec{v}_2 = (0,3,0,0)$, $\vec{v}_3 = (0,0,7,6)$, $\vec{v}_4 = (0,0,4,5)$

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 6 & 5 \end{vmatrix} = 2 \cdot 3(35 - 24) = 66 \neq 0.$$

So the four vectors (same number as $\dim(\mathbb{R}^4)$) are **linearly independent**, and hence do form a basis for $\mathbb{R}^4$.

Problem: **#13** Find a basis for the **subspace** of $\mathbb{R}^4$ which consists of vectors of the form (a,b,c,d) such that $a = 3c$ and $b = 4d$.

Can be written as... $\vec{v} = (3c, 4d, c, d)$

$= c(3,0,1,0) + d(0,4,0,1)$.

So let: $\vec{v}_1 = (3,0,1,0)$ and $\vec{v}_2 = (0,4,0,1)$.

And a basis is $\{\vec{v}_1, \vec{v}_2\}$.