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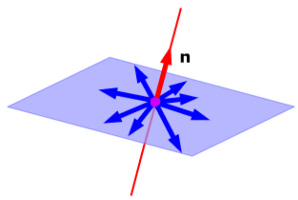
4.2: The Vector Space $\mathbb{R}^n$ and Subspaces

Dimensions $n > 3$, what are they good for?

Time, multiple particle systems, robotics,stock portfolios, local properties (pressure, temperature, color, velocity, etc.).

Vector addition/scalar multiplication in higher dimensions? Analogous to $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$.

Subspaces



Subspaces: Given vector space V (e.g., $\mathbb{R}^3$ in the image above), then W (the plane in the image) is a subset of V, and is called a subspace if:

- ♦ W is nonempty (it contains at least one vector),
- ♦ Given $\vec{u}, \vec{v} \in W$, we have $\vec{u} + \vec{v} \in W$, (W is closed under addition)
- ♦ Given $c \in \mathbb{R}$, we have $c\vec{u} \in W$. (W is closed under scalar multiplication)

And therefore $\vec{0} \in W$. Why?

Also, convince yourself that the x-axis and the y-axis (just the axes themselves, no other points), joined together as a subset of $\mathbb{R}^3$, does not constitute a subspace.

Solution Subspace Theorem: For $\mathbf{A}^{m \times n}$, the solution set of the homogeneous linear system

$\mathbf{A}\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

Proof: Let W denote the solution set of the system. If $\vec{u}$ and $\vec{v}$ are vectors in W, then $\mathbf{A}\vec{u} = \mathbf{A}\vec{v} = \vec{0}$.

Hence: $\mathbf{A}(\vec{u} + \vec{v}) = \mathbf{A}\vec{u} + \mathbf{A}\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

Thus the sum $\vec{u} + \vec{v}$ is also in W, and hence W is closed under addition.

Next, if $c \in \mathbb{R}$, then $\mathbf{A}(c\vec{u}) = c(\mathbf{A}\vec{u}) = c\vec{0} = \vec{0}$.

Thus $c\vec{u}$ is in W if $\vec{u}$ is in W. Hence W is also closed under scalar multiplication.

Therefore, W is a subspace of $\mathbb{R}^n$. ■

Nonhomogeneous System Solutions Theorem: The solution set of a nonhomogeneous system $\mathbf{A}\vec{x} = \vec{b}$ is **never** a subspace.

Proof: Let's do proof by contradiction. Let $\vec{u}$ be a solution in W, the set of solutions to the nonhomogeneous system above. And let us make the dubious assumption that W is a subspace.

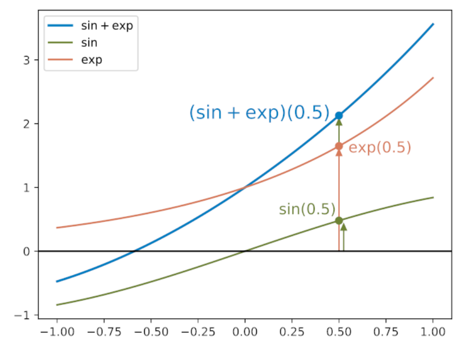
Let $c = 0 \in \mathbb{R}$. By closure under scalar multiplication, $c\vec{u} = 0\vec{u} = \vec{0}$ is a solution.

Therefore, $\mathbf{A}\vec{0} = \vec{0} = \vec{b}$. But we assumed the system was nonhomogeneous: $\vec{b} \neq \vec{0}$.

So we have a contradiction, and our assumption that the set of solutions W was a subspace was incorrect. ■

An alternative proof is provided in the book.

Function Spaces



Vector Space of Functions:

Let $F = \{$ all real valued functions $\}$; includes all polynomials, trig functions, exponentials, etc...

Observe that for $\mathbf{f}, \mathbf{g}$ in F, we have: $\mathbf{f}(x) + \mathbf{g}(x) = (\mathbf{f} + \mathbf{g})(x)$ and $c(\mathbf{f}(x)) = (c\mathbf{f})(x)$, which are also real valued functions.

The other properties of a vector space follow from the fact that these functions are real valued. The textbook proves one of the properties.

Video Tutorial (visually rich and intuitive): https://youtu.be/fNk_zzaMoSs

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Exercises

Problem: #16 For the following system of equations, find two solution vectors $\vec{u}$ and $\vec{v}$ (a basis) such that the **solution space** is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$.

$$x_1 - 4x_2 - 3x_3 - 7x_4 = 0$$
$$2x_1 - x_2 + x_3 + 7x_4 = 0$$

$$x_1 + 2x_2 + 3x_3 + 11x_4 = 0$$

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \overset{\text{trust me}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_2 = -s - 3t,$ and $x_1 = -s - 5t.$ So …

$$\vec{x} = (x_1, x_2, x_3, x_4) = \left(-s - 5t,\ \ -s - 3t,\ s,\ t\right)$$

$$= (-s, -s, s, 0) + (-5t, -3t, 0, t) = s\vec{u} + t\vec{v},\text{ where } \vec{u} = (-1, -1, 1, 0) \text{ and } \vec{v} = (-5, -3, 0, 1).$$

Problem: #22 Reduce the given system to echelon form to find a single solution vector $\vec{u}$ such that the solution space is the set of all scaler multiples of $\vec{u}$.

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 0, \\ 2x_1 + 7x_2 + 5x_3 - \ \ x_4 &= 0, \\ 2x_1 + 7x_2 + 4x_3 - 4x_4 &= 0. \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \\ 2 & 7 & 4 & -4 \end{bmatrix} \overset{\text{trust me}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus $x_4 = t$ is a parameter (a.k.a. free variable).

We solve for $x_1 = -6t,\ x_2 = 4t,$ and $x_3 = -3t.$ So,

$$\vec{x} = (x_1, x_2, x_3, x_4) = (-6t, 4t, -3t, t) = t\vec{u},\text{ where } \vec{u} = (-6, 4, -3, 1).$$

Problem: #29 Let $\mathbf{A}$ be an $n \times n$ matrix, $\vec{b}$ be a nonzero vector, and $\vec{x}_0$ be a solution vector to the system $\mathbf{A}\vec{x} = \vec{b}$. Show that $\vec{x_2}$ is another solution **if and only if** $(\Leftrightarrow)$ $\vec{x_2} - \vec{x}_0$ is a solution of the homogeneous system $\mathbf{A}\vec{y} = \vec{0}$.

We are given: $\mathbf{A}\vec{x}_0 = \vec{b}$.

Need to show that: $\mathbf{A}\vec{x_2} = \vec{b} \quad \Leftrightarrow \quad \mathbf{A}(\vec{x_2} - \vec{x}_0) = \vec{0}$.

Starting with the left assumption, and trying to show the thing on the right, we have:
$\mathbf{A}(\vec{x_2} - \vec{x}_0) = \mathbf{A}\vec{x_2} - \mathbf{A}\vec{x}_0 = \vec{b} - \vec{b} = 0.$ ✓

Going from right to left, we have: $\mathbf{A}(\vec{x_2} - \vec{x}_0) = \mathbf{A}\vec{x_2} - \mathbf{A}\vec{x}_0 = \mathbf{A}\vec{x_2} - \vec{b} = 0,$ therefore $\mathbf{A}\vec{x_2} = \vec{b}.$ **Q.E.D.**

Problem: #6. Assume W is the set of all vectors in $\mathbb{R}^4$ such that $x_1 = 3x_3$ and $x_2 = 4x_4$. Apply the theorems in this section to determine whether or not W is a subspace of $\mathbb{R}^4$.

$$W = \left\{\left(3c,\ 4d,\ c,\ d\right)\right\}.$$

First, note that the subspace is nonempty since $(3,4,1,1) \in W,$ where $c,d = 1$.

We arbitrarily choose two vectors from W by arbitrarily choosing four constant $c_1, d_1, c_2, d_2 \in \mathbb{R}$, giving us $\left(3c_1,\ 4d_1,\ c_1,\ d_1\right)$ and $\left(3c_2,\ 4d_2,\ c_2,\ d_2\right)$. We then test them for closure under addition:

$$\left(3c_1,\ 4d_1,\ c_1,\ d_1\right) + \left(3c_2,\ 4d_2,\ c_2,\ d_2\right)$$

$$= \left(3c_1 + 3c_2,\ 4d_1 + 4d_2,\ c_1 + c_2,\ d_1 + d_2\right)$$

$$= \left(3(c_1 + c_2),\ 4(d_1 + d_2),\ c_1 + c_2,\ d_1 + d_2\right) \in W$$

This is because it has the prescribed format $\left\{\left(3c,\ 4d,\ c,\ d\right)\right\}$, where $c = c_1 + c_2$ and $d = d_1 + d_2$.

Now to test scalar multiplication:

$$\alpha\left(3c_1,\ 4d_1,\ c_1,\ d_1\right) = \left(3\alpha c_1,\ 4\alpha d_1,\ \alpha c_1,\ \alpha d_1\right) \in W = \left\{\left(3c,\ 4d,\ c,\ d\right)\right\}$$

where $c = \alpha c_1$ and $d = \alpha d_1$.

Therefore, W is nonempty, closed under addition, and scalar multiplication, and is a subspace of $\mathbb{R}^4$. ■