Instructor: Jodin Morey moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

3.4: Matrix Operations

After assigning matrices some notation, like $\mathbf{A}, \mathbf{B}$, etc., it turns out we can start treating them as mathematical objects. What does this mean? It means we can define common operations like addition, subtraction, multiplication and using these operations turns out to be very useful.

But first let's define what it means for a matrix $\mathbf{A}$ to be equal to another matrix $\mathbf{B}$. It's probably what you would assume it should be, namely that they are the same size, and their elements are equal.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \neq \begin{bmatrix} 1 & 9 \\ 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The operations indicated below work for matrices of the same size. If your matrices are not of the same size, then some of the operations below will not work (particularly addition). Matrix multiplication has some size requirements which we will cover below.

Addition: $\mathbf{A} + \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}.$

Scalar Multiplication: $k\mathbf{A} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}.$

Negative of a matrix: $-\mathbf{A} = -1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.$

Therefore, subtraction is a kind of addition: $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.

Matrix Multiplication

Matrix multiplication is less intuitive. We wish to define it such that systems like:

$$\begin{aligned} 1x + 5y + 8z &= 0 \\ 2x + 6y + 9z &= 2 \\ 3x + 7y + 11z &= 4 \end{aligned},$$

with the coefficient matrix $\mathbf{A} = \begin{bmatrix} 1 & 5 & 8 \\ 2 & 6 & 9 \\ 3 & 7 & 11 \end{bmatrix}$, variable vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

and constant vector $\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$, can be written as $\mathbf{A}\vec{x} = \vec{b}$.

Matrix Multiplication is defined for $\mathbf{A}^{m \times n}$ and $\mathbf{B}^{n \times p}$, where the number of columns (n) in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$. The result is $\mathbf{A}^{m \times n}\mathbf{B}^{n \times p} = \mathbf{C}^{m \times p}$, so the inner indices "cancel."

$$\mathbf{A}^{3\times3}\mathbf{x}^{3\times1} = \begin{bmatrix} 1 & 5 & 8 \\ 2 & 6 & 9 \\ 3 & 7 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 5y + 8z \\ 2x + 6y + 9z \\ 3x + 7y + 11z \end{bmatrix} = \mathbf{C}^{3\times1}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} =$$

See animated in class

And similarly:

$$\mathbf{a}^{1\times3}\mathbf{c}^{3\times1} = \begin{bmatrix} 1 & 5 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 1 \cdot 1 + 5(-2) + 8 \cdot 3 = 15 = \mathbf{c}^{1\times1}$$ (dot product of vectors), and

$$\mathbf{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

When the size of the matrices are not compatible (as defined above), matrix multiplication is not defined.

Algebraic Properties: Identity and Zero Matrices

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{AI} = \mathbf{A}, \quad \mathbf{A} + \mathbf{0} = \mathbf{A}.$$

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ **Additive Commutative Property**

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ **Additive Associative Property**

$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ **Multiplicative Associative Property**

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$, and
$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ **Distributive Property**

However, AB is usually NOT equal to BA.

(i.e., matrices usually don't have "multiplicative commutivity")

If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, then notice that

$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, however $\mathbf{BA} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. So, $\mathbf{AB} \neq \mathbf{BA}$.

Column $\begin{bmatrix} a \\ b \end{bmatrix}$ and row $\begin{bmatrix} e & f \end{bmatrix}$ vector notation.

$\begin{bmatrix} a \\ b \end{bmatrix} = (a,b) \neq \begin{bmatrix} a & b \end{bmatrix}$. The parenthetical notation for $\begin{bmatrix} a & b \end{bmatrix}$ is $(a,b)^T$.

Video Tutorial (visually rich and intuitive, although you may have to pause and rewatch parts till it all sinks in):
https://youtu.be/XkY2DOUCWMU

Exercises

Problem: **#25** Let $\mathbf{A} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find $\mathbf{B}$ so that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$: (In other words, find the correct $\mathbf{B}$ so that $\mathbf{A}$ and $\mathbf{B}$ **DO** commute).

- First calculate, and then equate entries on the two sides of the equation $\mathbf{AB} = \mathbf{I}$.
- Then solve the resulting four equations for $a, b, c,$ and d.

The matrix equation $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ldots$

$$\begin{bmatrix} 5a+7c & 5b+7d \\ 2a+3c & 2b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$5a + 7c = 1$, $2a + 3c = 0$, $5b + 7d = 0$, $2b + 3d = 1$.

Then we solve for: $a = 3$, $b = -7$, $c = -2$, $d = 5$.

$$\mathbf{B} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}.$$

You can check your work by verifying that $\mathbf{BA} = \mathbf{I}$ as well.

$$\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 15 - 14 & 21 - 21 \\ -10 + 10 & -14 + 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

Problem 33: Find four different 2×2 matrices $\mathbf{A}$, with each main diagonal element either $+1$ or -1, such that $\mathbf{A}^2 = \mathbf{I}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$