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3.3: Reduced Echelon Matrices

Our task, try to convert:

$$\mathbf{A} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ a_{41} & a_{42} & a_{43} & b_4 \end{array}\right] \text{ into } \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1' \\ 0 & 1 & 0 & b_2' \\ 0 & 0 & 1 & b_3' \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{OR into } \left[\begin{array}{ccc|c} 1 & a_{12}' & 0 & b_1' \\ 0 & 0 & 1 & b_2' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \text{"Reduced Echelon Form"}$$

Unlike non-reduced echelon form, **it's unique**!!

Homogeneous Matrices

Definition: A linear system like **A** above is called **homogeneous** if the constants $b_1, b_2, \ldots, b_n$ on the right-hand side are all zeros.

Definition: A solution to a linear system like **A** above is called **trivial** if the solution consists of $x_1 = x_2 = \ldots = x_n = 0$.

You should convince yourself that the trivial solution is always a solution to a homogeneous system.

$$\mathbf{H} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{array}\right] \Rightarrow \begin{array}{c} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \\ a_{41}x + a_{42}y + a_{43}z = 0 \end{array} \Rightarrow \begin{array}{c} a_{11}\cdot 0 + a_{12}\cdot 0 + a_{13}\cdot 0 = 0 \\ a_{21}\cdot 0 + a_{22}\cdot 0 + a_{23}\cdot 0 = 0 \\ a_{31}\cdot 0 + a_{32}\cdot 0 + a_{33}\cdot 0 = 0 \\ a_{41}\cdot 0 + a_{42}\cdot 0 + a_{43}\cdot 0 = 0 \end{array}$$

Observe this is true irrespective of what the a_{ij} are.

What if you knew that **H** had at least one nontrivial solution?
Could you conclude something about how many solutions it has?

What about a homogeneous linear system **H** with more variables (columns) than equations (rows) how many solutions?

$$\mathbf{H} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array}\right]$$

$$\rightarrow \qquad \widetilde{\mathbf{H}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & 0 \\ 0 & 1 & c_{23} & 0 \end{array}\right] \text{ ??}$$

Nonhomogeneous Matrices

$$\mathbf{N} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ a_{41} & a_{42} & a_{43} & b_4 \end{array}\right], \text{ where at least one of the } b_i \neq 0.$$

Observe that after Gauss-Jordan elimination, **N** could become:

$$\widetilde{\mathbf{N}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & d_1 \\ 0 & 1 & c_{23} & d_2 \\ 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ where } d_3 \neq 0.$$

The third row reads: $0 \bullet x + 0 \bullet y + 0 \bullet z = d_3 \neq 0$. But this is impossible. We started with the assumption that the system had a solution, and then performed row manipulations in an attempt to find the solution. We ended up with the absurd statement that $0 = d_3 \neq 0$. So this tells us that we started with an absurd premise. In other words, our assumption that there was a solution was apparently absurd. The moral of the story is, if transforming your system to reduced echelon form results in a row which reads $0 = c \neq 0$, there are no solutions for the system.

Observe this couldn't happen with a homogeneous system, because the reduction process involves multiplying the zeros by constants, and then adding or subtracting them. But this still results in zeros! However, when you have nonzero constants, the reduction process may still result in a nonzero constant remaining, as above.

What about a nonhomogeneous linear system **N** with more variables (columns) than equations (rows) how many solutions?

$$\mathbf{N} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \end{array}\right] \quad \rightarrow \quad \widetilde{\mathbf{N}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & d_1 \\ 0 & 1 & c_{23} & d_2 \end{array}\right] \text{ ??}$$

$$\text{OR} \quad \rightarrow \quad \widetilde{\mathbf{N}} = \left[\begin{array}{ccc|c} 1 & c_{12} & c_{13} & d_1 \\ 0 & 0 & 0 & d_2 \end{array}\right] \text{ ??}$$

$n \times n$ Square Coefficient Matrices A

What if you had a homogeneous system **H** with the same number of variables (columns) as equations (rows)?

$$\mathbf{H} = [\mathbf{A}|\vec{0}] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right].$$

A is referred to as a 3×3 square *coefficient* matrix. More generally, we will see $n \times n$ square coefficient matrices.

We will be interested in discovering when such a system has only the trivial solution $x_1 = \ldots = x_n = 0$.

(Why? Because matrices like these can be used to easily solve *nonhomogeneous* equations easily)

If only the trivial solution, then **H** must be row equivalent to:

$$\widetilde{\mathbf{H}} = [\mathbf{I}|\vec{0}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$, where **I** is known as the 3×3 **identity matrix.**

That is, a matrix with ones on its **principal diagonal** (the one from upper left to lower right) and zeros elsewhere.

Similar Homogeneous System Theorem: Let **A** be a $n \times n$ matrix. Then the homogeneous system with coefficient matrix **A** has only the trivial solution if and only if **A** is row equivalent to the $n \times n$ identity matrix **I**.

Factoids:

- **Every homogeneous linear system with more variables (columns) than equations (rows) has infinitely many solutions**.
- **A non-homogeneous linear system with more variables (columns) than equations (rows) has infinitely many solutions or NO solutions**.

Exercises 

Problem: **#16** Find the reduced echelon form of the following matrix

$$\begin{bmatrix} 1 & 3 & 15 & 7 \\ 2 & 4 & 22 & 8 \\ 2 & 7 & 34 & 17 \end{bmatrix}$$

$$\xrightarrow{R2+(-2R1)} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & -2 & -8 & -6 \\ 2 & 7 & 34 & 17 \end{bmatrix} \xrightarrow{R3+(-2R1)} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & -2 & -8 & -6 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R2} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$$\xrightarrow{R3+(-R2)} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ This is echelon form.

Assuming this matrix was augmented, we could use back substitution from here, or …

$$\overset{R1+(-3R2)}{\rightarrow} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$x_3 = t_3, \qquad x_2 = -4t_3 + 3, \qquad x_1 = -3t_3 - 2$

Vector form: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3t_3 - 2 \\ -4t_3 + 3 \\ t_3 \end{bmatrix}$

$$= \begin{bmatrix} -3t_3 \\ -4t_3 \\ t_3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$= t_3 \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}.$$

Problem ≈ 15: Use the method of Gauss-Jordan elimination (reduced echelon form) to determine how many solutions the system has.

$$\begin{bmatrix} 2 & 2 & 4 & 2 \\ 1 & -1 & -4 & 3 \\ 2 & 7 & 19 & 7 \end{bmatrix} \Rightarrow \tfrac{1}{2}R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & -4 & 3 \\ 2 & 7 & 19 & 7 \end{bmatrix}$$

$$\Rightarrow R_3 + (-2R_1) \text{ and } R_2 + (-R_1) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -6 & 2 \\ 0 & 5 & 15 & 5 \end{bmatrix} \Rightarrow -\tfrac{1}{2}R_2 \text{ and } \tfrac{1}{5}R_3 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \qquad !?!?!$$

$$\Rightarrow R_3 + (-R_2) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow R_1 + (-R_2) \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

How many solutions? No solutions.

Problem: **#39** Consider a homogeneous (what's this?) system of three equations in three unknowns. Suppose that the third equation is the sum of a multiple of the first equation and a multiple of the second equation. Show that the system has a nontrivial solution (what's this?).

It is given that the augmented coefficient matrix of the homogeneous 3 x 3 system has the form…

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ pa_1 + qa_2 & pb_1 + qb_2 & pc_1 + qc_2 & 0 \end{bmatrix}.$$

Upon subtracting both p times row 1 and q times row 2 from row 3, we get the matrix…

$$\Rightarrow R_3 + (-pR_1) \text{ and } R_3 + (-qR_2) \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

... corresponding to two homogeneous linear equations in three unknowns. Hence there is at least one free variable, and thus the system has a nontrivial family of (infinitely many) solutions.

Problem 32 **Show that the** 2×2 **matrix**: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **is row equivalent to the** 2×2 **identity matrix**, **provided that** $ad - bc \neq 0$.

Observe that the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

HINTS: For them to be row equivalent, we need to be able to transform one into the other.

Try to do row operations on **A**, to get it into some form which looks like **I**.

Observe what restrictions on a, b, c, d arise as you do this (do not divide by zero!).

What restrictions flow from $ad - bc \neq 0$? Obviously a, b, c, d can't all be zero, can you provide less restrictive restrictions?