

10/29/2020

MATH 2243: Linear Algebra & Differential Equations

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2.3: Acceleration-Velocity Models

Recall that (when neglecting air resistance) the force ($F = ma$) exerted on a mass under gravity can be notated: $F_G = m \frac{dv}{dt} = -mg$ (where $g \approx 9.8 \text{ m/sec}^2 \approx 32 \text{ ft/sec}^2$ represents the acceleration of an object under the force of gravity at the surface of the Earth).

In real life, we have air resistance as an additional force, so to get the total force F_{total} , we must add in the additional air resistance force term F_R , so: $F_{total} = F_G + F_R$.

Due to various complexities (shape of the object, temperature, humidity, etc.), the force of air resistance (which also depends on velocity v) is written to allow us some flexibility: $F_R = \pm kv^p$ where k and p are constants with $1 \leq p \leq 2$. It suffices for our purposes to study instances when $p = 1$ or 2 . So, $F_{total} = -mg \pm kv^p$.

Case: $p = 1$

Also known as "air resistance is proportional to the velocity."

$F_R = -kv = -k \frac{dy}{dt}$, where y is the distance from the ground.

Note that whether an object is falling or rising, the force of air resistance (F_R)

has an opposite sign compared to the velocity ($\frac{dy}{dt}$). As things rise, air

resistance pushes down ($\frac{dy}{dt} > 0$ points up. So $F_R < 0$ points down, same as gravity).

And as things fall, air-resistance pushes up ($\frac{dy}{dt} < 0$, so $F_R > 0$, opposite from gravity).

So, $F_{tot} = F_G + F_R = -mg - kv$.

(confirm in your mind that $-kv$ is a positive number if an object is falling, but negative when rising)

Definitions: $\rho = \frac{k}{m}$ is the **drag coefficient**,

$a_R = \frac{F_R}{m} = -\frac{k}{m} v^p = -\rho v^p$ is **acceleration due to air resistance**,

and $|v_\tau| = \frac{g}{\rho}$ is the **terminal speed for $p = 1$** .

Case: $p = 2$

Also known as "air resistance is proportional to the **square** of the velocity ($a_R = \pm \rho v^2$)."

Recalling that we have $m \frac{dv}{dt} = -mg + F_R$, and that we need to keep the sign of air resistance (F_R) opposite the direction of velocity v , we get:

$$F_R = -kv|v|, \text{ and } a_{total} = \frac{dv}{dt} = -g - \rho v|v|.$$

Upward Motion: happens when $v > 0$:

$$\frac{dv}{dt} = -g - \rho v^2 \quad (\text{Eq. 12})$$

$$\text{Velocity: } v(t) = \sqrt{\frac{g}{\rho}} \tan\left(\tan^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right) - t \sqrt{\rho g}\right), \quad (\text{Eq. 13})$$

$$\text{Position: } y(t) = y_0 + \frac{1}{\rho} \ln \left| \frac{\cos\left(\tan^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right) - t \sqrt{\rho g}\right)}{\cos\left(\tan^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right)\right)} \right|. \quad (\text{Eq. 14})$$

Downward Motion: $v < 0$...

$$\frac{dv}{dt} = -g + \rho v^2 \quad (\text{Eq. 15})$$

$$\text{Velocity: } v(t) = \sqrt{\frac{g}{\rho}} \tanh\left(\tanh^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right) - t \sqrt{\rho g}\right), \quad (\text{Eq. 16})$$

$$\text{Position: } y(t) = y_0 - \frac{1}{\rho} \ln \left| \frac{\cosh\left(\tanh^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right) - t \sqrt{\rho g}\right)}{\cosh\left(\tanh^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right)\right)} \right|. \quad (\text{Eq. 17})$$

Note: $|v_\tau| = \sqrt{\frac{g}{\rho}} = \sqrt{\frac{mg}{k}}$ is the **terminal speed** for $p = 2$.

Universal Gravitational Force:

Assume two bodies have masses M and m , and are r distance apart.

Then between them we have the force:

$$F_G = \frac{GMm}{r^2}, \text{ where } G \approx 6.6726 \times 10^{-11} \text{ N} \cdot (\text{m/kg})^2.$$

Applying a constant thrust (acceleration) a_T to slow down the landing of a spacecraft...

$$a = \frac{d^2r}{dt^2} = a_T - \frac{F_G}{m} = a_T - \frac{GM}{r^2}, \text{ where } M \text{ is the mass of the planet/moon,}$$

m is the mass of the spacecraft, and r is the radius to the center of the planet.

This is a special type of DEQ, where there is a simplification we can make to help solve it.

Notice that first derivative $\frac{dr}{dt} = v$ can be characterized as just a function of r and not t .

In other words: $v(r)$ (is autonomous).

So, the DEQ in the previous paragraph can be changed into a first order DEQ

by recognizing that $\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = \frac{dv}{dr} v$ (notice the use of the chain rule).

Therefore, we can instead solve the simplified seperable DEQ ($v \frac{dv}{dr} = a_T - \frac{GM}{r^2}$) to find:

$$\frac{1}{2} v^2 = a_T r + \frac{GM}{r} + C, \text{ (try calculating this yourself.)}$$

We can now plug in our various given quantities and any initial conditions to determine our arbitrary constant C .

Escape Velocity:

$v_0 = \sqrt{\frac{2GM}{r}}$ is the **escape velocity** on any planet/moon,

where r is distance from the center of the planet/moon to its surface, and M is the mass of the planet/moon.

Problem: #3 Suppose that a motorboat is moving at 40 ft/sec when its motor suddenly quits, and then 10 sec later the boat has slowed to 20 ft/sec. Assume, that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

Eventually, what do we want to calculate?

$$x(\infty) := x(t) \text{ as } t \rightarrow \infty.$$

What is the equation for "resistance being proportional to velocity?"

$$F_R = -kv.$$

Dividing by mass, we have: $a = v' = -\rho v$.

How should we write the initial conditions?

$$v(0) = 40; \quad v(10) = 20, \quad x(0) = 0.$$

$$\int \frac{1}{v} dv = -\rho \int dt, \quad \ln|v| = -\rho t + c, \quad v = e^{-\rho t + c} = e^c e^{-\rho t}, \text{ where } e^c > 0.$$

$$40 = e^c, \quad v(t) = 40e^{-\rho t},$$

$$20 = 40e^{-10\rho}, \quad e^{-10\rho} = \frac{1}{2}, \quad -10\rho = \ln \frac{1}{2} = -\ln 2, \quad \rho = \frac{\ln 2}{10}.$$

$$\int dx = 40 \int e^{-\rho t} dt, \quad x = -\frac{40}{\rho} e^{-\rho t} + C,$$

$$0 = 40\left(-\frac{1}{\rho}\right)e^0 + C, \quad C = \frac{40}{\rho}.$$

$$x(t) = \frac{40}{\rho}(1 - e^{-\rho t})$$

"How far will the boat coast in all?"

$$x(\infty) = \lim_{t \rightarrow \infty} \frac{40}{\rho}(1 - e^{-\rho t}) = \frac{40}{\rho} \approx 577 \text{ ft.}$$

Problem: #17 Consider a crossbow bolt (from the example in the book) shot straight upward from the ground ($y = 0$), at time $t = 0$ with initial velocity $v_0 = 49 \text{ m/s}$. Use $g = 9.8 \text{ m/s}^2$ and $\rho = 0.0011$ in Eq. (12). Then show that the bolt reaches its maximum height of about 108.47 m in about 4.61 s.

Recall:

$$\frac{dv}{dt} = -g - \rho v^2 \quad (\text{Eq. 12})$$

$$\frac{dv}{dt} = -9.8 - 0.0011v^2$$

$$\int \frac{dv}{9.8+0.0011v^2} = -\int dt \quad (*)$$

Recall that: $(\tan^{-1}v)' = \frac{1}{1+v^2}$.

So the left-hand side of (*) should change to:

$$\int \frac{\frac{1}{9.8}}{1+(0.010595v)^2} dv = \frac{1}{(9.8)} \int \frac{1}{1+(0.010595v)^2} dv.$$

u-sub: $u = 0.010595v, \quad du = 0.010595dv$

$$\frac{1}{9.8(0.010595)} \int \frac{1}{1+u^2} du = \frac{1}{9.8(0.010595)} \tan^{-1}(0.010595v) + C$$

$$\Rightarrow \tan^{-1}(0.010595v) = -0.103827t + C.$$

$v(0) = 49$ implies $C + 0 = \tan^{-1}(0.010595 \cdot 49) = 0.478854$.

So, $\tan^{-1}(0.010595v) = -0.103827t + 0.478854$

$$\Rightarrow 0.010595v = \tan(0.478854 - 0.103827t),$$

$$\Rightarrow v(t) = 94.3841 \tan(0.478854 - 0.103827t).$$

Or, if we just plug in our known values into Equation 13:

Velocity: $v(t) = \sqrt{\frac{g}{\rho}} \tan\left(\tan^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right) - t \sqrt{\rho g}\right)$, we find we get the same answer.

"Show that the bolt reaches its maximum height of about 108.47 m in about 4.61 s."

$$\int dy = 94.3841 \int \tan(0.478854 - 0.103827t) dt$$

$$\Rightarrow y = 94.3841 \left[\frac{1}{0.103827} \ln |\cos(0.478854 - 0.103827t)| \right] + C$$

Plugging in our initial condition $y = 0$ at $t = 0$:

$$0 = 94.3841 \left[\frac{1}{0.103827} \ln(\cos(0.478854)) \right] + C$$

$$\Rightarrow C \approx 108.5.$$

$$\text{So, } y(t) \approx 94.38 [909 \ln |\cos(0.4788 - 0.1038t)|] + 108.5$$

How do we find out the time the bolt is at its maximum height?

Recall from above: $v(t) = 94.38 \tan(0.4788 - 0.1038t)$

$$0 = 94.38 \tan(0.4788 - 0.1038t), \quad 0.4788 - 0.1038t = \tan^{-1}(0) = 0$$

$$\Rightarrow 0.4788 - 0.1038t = 0,$$

$$\Rightarrow t = \frac{0.4788}{0.1038} \approx 4.61 \text{ sec.}$$

$$\text{So maximum height is: } y(4.61) \approx 94.38 [909 \ln |\cos(0.4788 - (0.1038)(4.61))|] + 108.5$$

$$\approx 108.47 \text{ m.}$$

Problem: #22

Suppose that $\rho = 0.075$ in Eq. 15 ($\frac{dv}{dt} = -g + \rho v^2$ in fps units, with $g = 32 \text{ ft/s}^2$) for a paratrooper falling with parachute open. If he jumps from an altitude of 10,000 feet and opens his parachute immediately, what will be his terminal speed? How long will it take him to reach the ground?

By an integration (or Eq. 13: $v(t) = \sqrt{\frac{g}{\rho}} \tan\left(\tan^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right)$), the solution of the initial value problem $v' = -32 + 0.075v^2$, $v(0) = 0$ is $v(t) = -20.666 \tanh(1.54919t)$.

And since $\tanh(t) \rightarrow -1$ as $t \rightarrow -\infty$, we have:

$$\text{terminal speed} = |v_\tau| = \left| -\sqrt{\frac{g}{\rho}} \right| = 20.666 \text{ ft/sec.}$$

"How long will it take him to reach the ground?"

$$\text{Position: } y(t) = y_0 - \frac{1}{\rho} \ln \left| \frac{\cosh\left(\tanh^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right) - t\sqrt{\rho g}\right)}{\cosh\left(\tanh^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right)\right)} \right|. \quad (\text{Eq. 17})$$

With $y(0) = 10,000$, we get $y(t) = 10,000 - 13.333 \ln(\cosh(1.54919t))$.

We solve for $y(t) = 0$, and get $t = 484.57$ sec.

$$\frac{484.57 \text{ sec}}{60 \text{ min/sec}} \approx 8.0762 \text{ min. And } 0.0762 \text{ min} \cdot 60 \frac{\text{sec}}{\text{min}} \approx 4.57 \text{ sec.}$$

Thus the descent takes about 8 min 5 sec.

Problem: #24

The mass of the sun is 329,320 times that of the earth and its radius is 109 times the radius of the Earth.

a) To what radius would the earth have to be compressed in order for it to become a black hole? The escape velocity from its surface would need to be equal to 3×10^8 m/s (the speed of light!).

$$\text{Recall: Escape Velocity is } v_0 = \sqrt{\frac{2GM}{R}}.$$

Also recall: that the gravitational constant $G = 6.6726 \times 10^{-11} \text{ (kg} \cdot \text{m} \cdot \text{s}^2) \cdot (\text{m/kg})^2$, and the mass of the earth is $M = 5.975 \times 10^{24} \text{ kg}$.

$$\text{So, } \sqrt{\frac{2GM}{R}} = 3 \times 10^8 \text{ m/s, and}$$

$$R = \frac{2GM}{(3 \times 10^8 \text{ m/s})^2} = \frac{2(6.6726 \times 10^{-11} \text{ (kg} \cdot \text{m} \cdot \text{s}^2) \cdot (\text{m/kg})^2)(5.975 \times 10^{24} \text{ (kg)})}{(3 \times 10^8 \text{ m/s})^2}$$

$$R = \frac{79.6975344 \times 10^{13}}{9 \times 10^{16}} = 8.85973 \times 10^{-3} \text{ meters, or about } 0.88 \text{ cm.}$$

b) Repeat part a) with the sun in place of the earth.

$$\text{So, } \sqrt{\frac{2G(329,320 \cdot M)}{R}} = c, \text{ then } R = \frac{2 \cdot 329,320 \cdot GM}{(3 \times 10^8 \text{ m/s})^2} = \frac{658,320(6.6726 \times 10^{-11} \text{ (kg} \cdot \text{m} \cdot \text{s}^2) \cdot (\text{m/kg})^2)(5.975 \times 10^{24} \text{ (kg)})}{(3 \times 10^8 \text{ m/s})^2}$$

$$R = \frac{26246418.5412 \times 10^{13}}{9 \times 10^{16}} = 2916268.7268 \times 10^{-3} \text{ meters, or about } 2.92 \text{ km (1.8 miles!!).}$$