MATH 2243: **Linear Algebra & Differential Equations**

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1.5: Linear First-Order Equations

Integrating Factor

We previously solved DEQs like $\frac{dy}{dx} = 2xy$ by multiplying both sides by an **integrating factor**.

That is, an expression that leaves both sides of the equation resembling some derivative.

In this case, for $\frac{1}{y}\frac{dy}{dx} = 2x,$ we see that the left-hand side is $\frac{d}{dx}(\ln y)$, and the right hand side is $\frac{d}{dx}(x^2)$.

It is this fact which allows us to integrate both sides (with respect to the same variable) and solve the DEQ.

Linear First-Order Eq

How to solve $\frac{dy}{dx} + P(x)y = Q(x),$ when P, Q are continuous?

We can't just integrate right away, and the integrating factor is not so obvious here.

It turns out the integrating factor for this type of DEQ is: $\rho(x) := e^{\int P(x)dx}$.

To see this, let's multiply both sides:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}. \qquad (*)$$

Now is the tricky part is to recognize that the left-hand side of $(*)$ is a result of taking a derivative with the product rule. Recall that $\frac{d}{dx}\left[\int P(x)dx\right] = P(x),$ and so:

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}y' + \frac{d}{dx}\left(e^{\int P(x)dx}\right)y = e^{\int P(x)dx}y' + e^{\int P(x)dx}\frac{d}{dx}\left(\int P(x)dx\right)y$$

$$= e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y.$$

So $(*)$ can be rewritten as: $\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = Q(x)e^{\int P(x)dx}$.

And integrating both sides with respect to x, we find: $e^{\int P(x)dx}y = \int Q(x)e^{\int P(x)dx}dx + C.$

And solving for y, we find the solution to the DEQ: $\quad y = \frac{1}{\rho}\left(\int \rho Q(x)dx + C\right)$.

Are any of these first order DEQs **linear**?: $\frac{dy}{dx} + y^2 + c = 0, \qquad \frac{dy}{dx} = y + \frac{x}{y} + c, \qquad \frac{dy}{dx} = y^{\frac{1}{5}}$.

Linear First-Order Eq Solution Theorem

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem: $\frac{dy}{dx} + P(x)y = Q(x)$, with initial conditions $y(x_0) = y_0$ has a unique solution $y(x)$ on the *entire interval* I, given by the formula above.

No singular solutions (because $\rho \neq 0$).

Exercises 💪

Problem: #**24** Find the general solution of the differential equation: $(x^2+4)y' + 3xy = x$.
Using the initial condition: $y(0) = 1$, find the corresponding particular solution.

First, calculate the integrating factor:

$y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}$ (observe that $x^2 + 4 \neq 0$, for any x, so we don't need to worry about dividing)

$\rho = \exp\left(\int \frac{3x}{x^2+4}dx\right)$,

$u = x^2 + 4, \qquad du = 2xdx$

$\rho = \exp\left[\frac{3}{2}\int \frac{1}{u}du\right]$,

$\rho = \exp[\frac{3}{2}\ln(x^2+4)]$, (why no $+C$?)

$\rho = e^{\ln\left((x^2+4)^{\frac{3}{2}}\right)} = (x^2+4)^{\frac{3}{2}}$. Now what?

Multiply both sides of the equation ($y' + \frac{3x}{x^2+4}y = \frac{x}{x^2+4}$) by the integrating factor:

$\rho\left(y' + \frac{3x}{x^2+4}y\right) = \rho\frac{x}{x^2+4}$ or $(x^2+4)^{\frac{3}{2}}\left[y' + \frac{3x}{x^2+4}y\right] = (x^2+4)^{\frac{3}{2}}\left[\frac{x}{x^2+4}\right]$.

Simplifying, we have: $(x^2+4)^{\frac{3}{2}}y' + 3x(x^2+4)^{\frac{1}{2}}y = x(x^2+4)^{\frac{1}{2}}$.

Next, recognize that the left-hand side of the equation

is the derivative of the following product: $(x^2+4)^{\frac{3}{2}}y$, which is just ρy.

So, $(\rho y)' = x(x^2+4)^{\frac{1}{2}}$

(and remember our goal is to discover what y is!)

Therefore, integrating the equation:

LHS: ρy or $y(x^2+4)^{\frac{3}{2}}$

RHS: $\int\left[x(x^2+4)^{\frac{1}{2}}\right]dx$.

Using u substitution: $u = x^2+4$, $du = 2xdx$, we have:

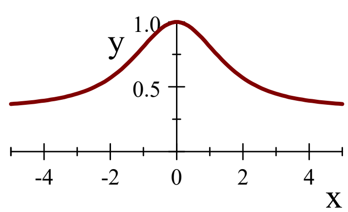
$$y(x^2+4)^{\frac{3}{2}} = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}+2C\right) = \frac{1}{3}(x^2+4)^{\frac{3}{2}}+C$$

$$y = \frac{1}{3}+C(x^2+4)^{-\frac{3}{2}}$$

"Using initial condition: $y(0) = 1$, find corresponding particular solution"

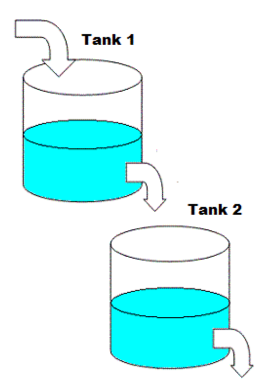
$1 = \frac{1}{3}+C(0^2+4)^{-\frac{3}{2}} = \frac{1}{3}+\frac{1}{8}C, \qquad \Rightarrow \qquad C = \frac{16}{3}$.

So $y(x) = \frac{1}{3}+\frac{16}{3}(x^2+4)^{-\frac{3}{2}}$.



$\frac{1}{3}+\frac{16}{3}(x^2+4)^{-\frac{3}{2}}$

Problem: #38



Consider the cascade of two tanks shown, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) representing the volumes of solution (salt water) in the two tanks. Dissolved in the water of each tank is 50 lb of salt. The three flow rates indicated in the figure are each 5 $gal/\min$, with pure water flowing into tank 1.

a) Find the amount of salt in tank 1 at time t, represented as $x(t)$.

$\frac{dx}{dt} = [IN]-[OUT] = 0-5\bullet\frac{x}{100} = -\frac{x}{20}$.

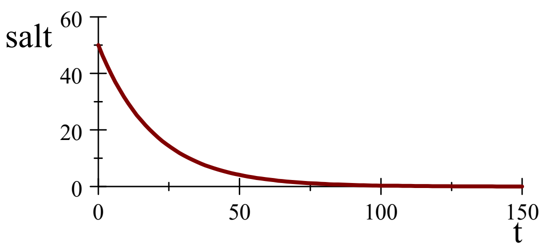
$\int \frac{d}{dt} \ln|x| dt = -\frac{1}{20} \int dt \qquad \Rightarrow \qquad \ln|x| = -\frac{t}{20} + c$

$\Rightarrow \qquad x = Ce^{-\frac{t}{20}}$, where $C > 0$. (because a nonzero amount of salt is always positive)

Any initial conditions?

$x(0) = 50$, so $50 = Ce^0$ and $C = 50$.

Therefore, $x(t) = 50e^{-\frac{t}{20}}$.



$50e^{-\frac{t}{20}}$

b) Suppose that $y(t)$ represents the amount of salt in tank 2 at time t. Solve for $y(t)$.

$\frac{dy}{dt} = [IN] - [OUT]$

$= \frac{x}{20} - \frac{5y}{200}$

$= \frac{50e^{-\frac{t}{20}}}{20} - \frac{5y}{200} = \frac{5}{2}e^{-\frac{t}{20}} - \frac{y}{40}.$

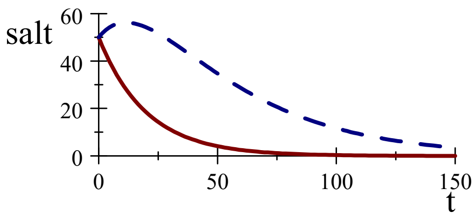
$\frac{dy}{dt} + \frac{y}{40} = \frac{5}{2}e^{\frac{-t}{20}} \qquad \Rightarrow \qquad \rho = e^{\frac{1}{40}\int dt} = e^{\frac{t}{40}}$

$y = e^{\frac{-t}{40}} \frac{5}{2} \int e^{\frac{t}{40}} e^{\frac{-t}{20}} dt = \frac{5}{2} e^{\frac{-t}{40}} \int e^{\frac{-t}{40}} dt$

$y = \frac{5}{2} e^{\frac{-t}{40}} (-40) e^{\frac{-t}{40}} + e^{\frac{-t}{40}} C = -100e^{\frac{-t}{20}} + e^{\frac{-t}{40}} C$

When $y(0) = 50 = -100e^0 + Ce^0$. $\qquad$ So, $C = 150$.

And $y = -100e^{\frac{-t}{20}} + 150e^{\frac{-t}{40}}$.



$50e^{\frac{-t}{20}}$ and $-100e^{\frac{-t}{20}} + 150e^{\frac{-t}{40}}$

c) Finally, find the maximum amount of salt ever in tank 2.

$$y'(t) = \left(-\tfrac{1}{20}\right)\left(-100e^{\frac{-t}{20}}\right) + \left(-\tfrac{1}{40}\right)\left(150e^{\frac{-t}{40}}\right) = 5e^{\frac{-t}{20}} - \tfrac{15}{4}e^{\frac{-t}{40}} = 0$$

$4e^{\frac{-2t}{40}} - 3e^{\frac{-t}{40}} = 0,$ $\qquad 4e^{\frac{-t}{40}} - 3 = 0$ $\qquad$ (multiplied through by $e^{\frac{t}{40}}$)

$e^{\frac{-t}{40}} = \frac{3}{4}$ $\qquad -\frac{t}{40} = \ln\frac{3}{4}$

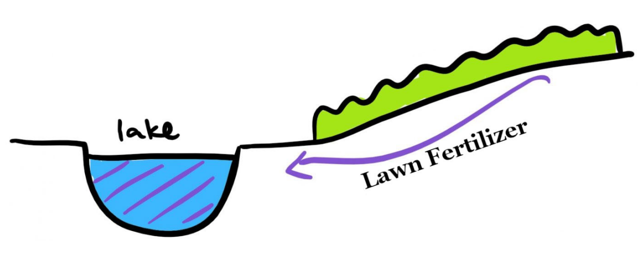
$t = -40\ln\frac{3}{4} \approx 11.51$ min.

"Find the maximum amount of salt"

$y_{\max} = y(11.51) = 56.25\ lb.$

Note: Shortcut to explicit solution to DEQ: $y = \frac{1}{\rho}\int \rho\, Q dx$.

Problem: **#34** Consider a lake with a volume of 8 billion cubic feet (ft^3) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft^3 of water with a pollutant concentration of 0.05% and an equal daily outflow of the (well-mixed) water in the lake. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?



Let $x(t)$ denote the amount of pollutants in the lake after t days.

We set up the differential equation in infinitesimal form by writing...
$\frac{dx}{dt} = [IN] - [OUT]$.

IN = (Concentration) $\times$ (rate of input) $\qquad = \qquad ?$

$IN = (0.0005)(500).$ (counting in millions)

$OUT =$ (Concentration) $\times$ (rate of input) $=$?

The volume of the lake is $8,000\ mft^3$.

$OUT = \frac{x}{8000} \cdot 500.$

$\frac{dx}{dt} = [IN] - [OUT] = (0.0005)(500) - \frac{x}{8000} \cdot 500,$

which simplifies to $\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}$, or $\frac{dx}{dt} + \frac{1}{16}x = \frac{1}{4}$. What's next?

First, calculate the integrating factor: $\rho = e^{\int \frac{1}{16}dt} = e^{\frac{t}{16}}$. Then, ...

Multiply both sides of the equation by the integrating factor

$e^{\frac{t}{16}}\left(\frac{dx}{dt} + \frac{1}{16}x\right) = \frac{1}{4}e^{\frac{t}{16}}$

Next, recognize the left-hand side is the derivative of the product, $xe^{\frac{t}{16}}$.

So, integrate the equation: $D_t\left(xe^{\frac{t}{16}}\right) = \frac{1}{4}e^{\frac{t}{16}}$, to get:

$xe^{\frac{t}{16}} = \frac{1}{4}\int e^{\frac{t}{16}}dt = \frac{1}{4}\left(e^{\frac{t}{16}} \cdot 16\right) + C = 4e^{\frac{t}{16}} + C.$ $x(t) = 4 + Ce^{-\frac{t}{16}}$ And then...

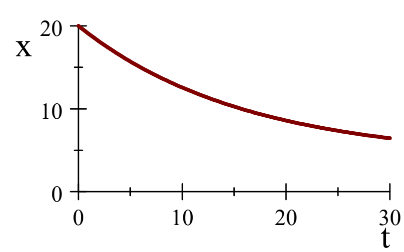
The initial amount $x(0)$ of pollutants is...

$x_0 = (0.25\%)(8000) = (0.0025)(8000) = 20\ mft^3.$

Plug-in our initial conditions: $x(0) = 4 + Ce^0 = 4 + C = 20,$

$\Rightarrow \quad C = 16.$

$x(t) = 4 + 16e^{-\frac{t}{16}}.$ Are we done?



$4 + 16e^{-\frac{t}{16}}$

We want to know when $x(t) = (0.10\%)(8000) = (0.0010)(8000) = 8\ mft^3.$

$8 = 4 + 16e^{-\frac{t}{16}}, \quad \Rightarrow \quad \frac{1}{4} = e^{-\frac{t}{16}}, \quad \Rightarrow \quad e^{\frac{t}{16}} = 4$

$\Rightarrow \quad \ln 4 = \frac{t}{16}, \quad \Rightarrow \quad t = 16\ln 4 \approx 22.2\ days.$

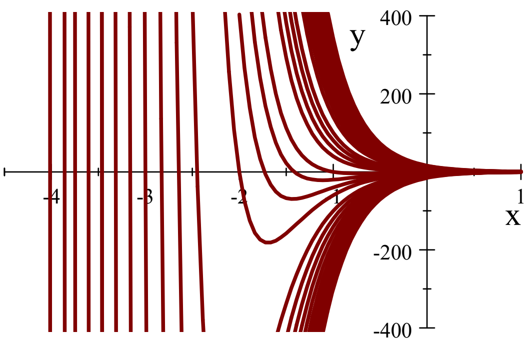
Problem: #3 Find the general solution of the differential equation: $y' = 2xe^{-3x} - 3y$.

$y' + 3y = 2xe^{-3x}$

$\rho = e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}$. (observe that $P(x) = 3$ NOT $3y$)

$y = \frac{1}{\rho}\int \rho Q(x)dx = e^{-3x}\int(e^{3x} \cdot 2xe^{-3x})dx$

$= e^{-3x}\int 2xdx = e^{-3x}(x^2 + c)$.



$e^{-3x}(x^2 + c)$ for various c