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MATH 2243: **Linear Algebra** & **Differential Equations**

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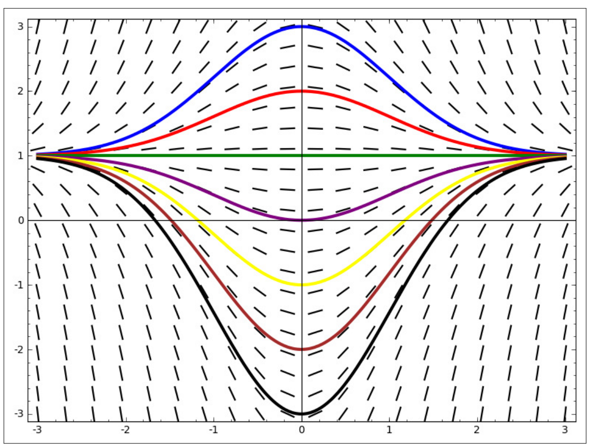
1.3: Linear Systems and Matrices

How to solve: $\frac{dy}{dx} = f(x,y)$? $\int \frac{dy}{dx} dx = \int f(x,y) dx$?!?

$f(x,y)$ may involve unknown function $y(x)$, and therefore integrating $f(x,y)$ with respect to x may not be possible in this way. We need another method!

While not all differential equations are **analytically** solvable (we can't solve them exactly), we can nonetheless draw **slope fields** for equations $y' = f(x,y)$. Do this by choosing any point (x,y), plug these values into $f(x,y)$ and this gives you a slope, then graph a short line at (x,y) having the slope $y' = f(x,y)$. Repeat as needed. See the short slope marks in the graph below, and some particular solutions drawn through them.



Drawing paths in the plane that are parallel to the nearby slope marks (as in the graph above) gives you a **solution curve**, a curve representing a solution to your DEQ.

WolframAlpha.com

Type in DEQs (for example if $y' = f(x,y)$) using the syntax:

"stream plot (1, f(x,y)), x = -1..5, y = -1..5"

Existence and Uniqueness of Solutions:

No one wants to search for something that doesn't exist!
Example: $y' = \frac{1}{x}$, $y(0) = 0$ cannot be solved because:
$\int \frac{1}{x} dx = \ln|x| + c$, which is not defined when $x = 0$.

Criteria:

Given: $\frac{dy}{dx} = f(x,y)$, $y(a) = b$.
Assume $f(x,y)$ and $\frac{\partial}{\partial y}[f(x,y)]$ are continuous on some rectangle R (containing (a,b) in its interior).
[For a concrete example, let's say they are continuous in the rectangle $x \in (2,3)$ and $y \in (-1,4)$

where $(a,b) = \big(2.6,\ \ 1.0\big)$).]
Then, **Theorem 1.3** from the book guarantees the existence & uniqueness of a **local** solution in R, in some interval $x \in I$ (still containing a), but possibly smaller than the width of R.
[Continuing our previous concrete example, the interval I might turn out to be $x \in (2.5, 2.8)$, and observe $a = 2.6$ is still in this interval).

Particularly, continuity of $f(x,y)$ guarantees existence on I,
and continuity of $\frac{\partial}{\partial y}[f(x,y)]$ guarantees uniqueness of that solution.

In addition to being continuous, if $\frac{\partial}{\partial y}[f(x,y)]$ is also **bounded** for all x *and* y,
then **global** existence/uniqueness (on R) of a solution y is guaranteed.
But which functions are bounded?
Polynomials are not, but the following functions are: $\sin x,\ \cos x,\ \frac{1}{1+x^2},\ e^{-|x|^2}$, etc.

Another way to ensure that you have global existence (on R) is to check if the equation is a **linear first-order** DEQ (i.e. $ay' + by + c = d$). *ALL* such differential equations have global existence of solutions.

Negative results from the above tests **tell you nothing**. In other words, even if these tests fail, the solution may still have existence/uniqueness/globalness! (you may have to think abou this one.)

Problem: #18

Determine whether existence of at least one solution of the initial value problem
$y\frac{dy}{dx} = x - 1;\ y(1) = 0$ is guaranteed. If so, then is uniqueness of that solution also guaranteed?

$f(x,y) = \frac{x-1}{y}$, $\quad \frac{\partial}{\partial y}f = \frac{-(x-1)}{y^2}$ $\quad$ Continuous near $(1,0)$?

Neither $f(x,y) = \frac{x-1}{y}$ nor $\frac{\partial}{\partial y}f = \frac{-(x-1)}{y^2}$ is continuous near $(1,0)$, so the existence-uniqueness theorem guarantees nothing.

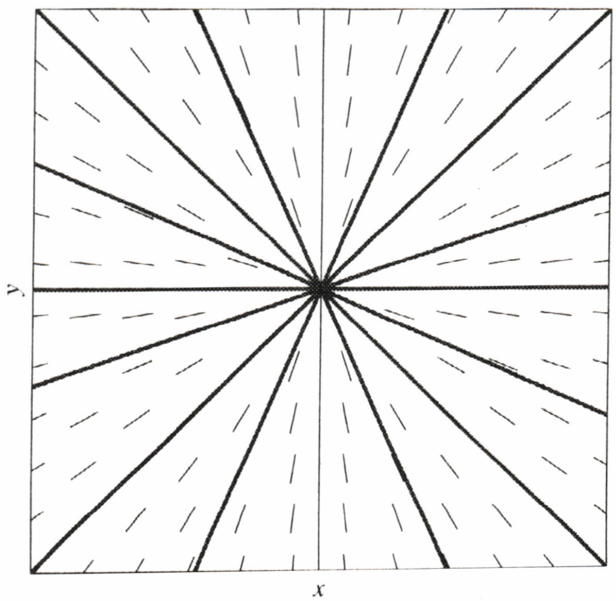
Problem: #28

This problem illustrates the fact that, if the hypotheses of **Theorem 1.3** above are NOT satisfied, then the initial value problem $y' = f(x,y),\ y(a) = b$ may have either

- no solutions (no existence)
- finitely many solutions, or (existence, possibly uniqueness)
- infinitely many solutions. (no uniqueness)

Verify that if k is a constant, then the function $y(x) = kx$ satisfies the differential equation: $xy' = y$ for all x. Note that $y' = \frac{y}{x}$ is not continuous at $(0,b)$. So Theorem 1.3 is NOT satisfied.

Construct a slope field, and several of these straight-line solution curves.



Then determine how many different solutions the initial value problem: $xy' = y,\ y(a) = b$ has for various (a, b). — one, none, or infinitely many.

The initial value problem has...

- a unique solution off the y-axis where $a \neq 0$;
- infinitely many solutions through the origin where $a = b = 0$;
- no solution if $a = 0$ but $b \neq 0$ (so the point (a, b) lies on the positive or negative y-axis).

Problem: **#20**

Determine whether existence of at least one solution of the initial value problem $\frac{dy}{dx} = x^2 - y^2;\ y(0) = 1$ is guaranteed and, if so, whether uniqueness of that solution is guaranteed.

Both $f(x, y) = x^2 - y^2$ and $\frac{\partial f}{\partial y} = -2y$ are continuous near $(0, 1)$ (and everywhere!), so the theorem guarantees both existence and uniqueness of a solution in some (any!) rectangle containing $x = 0$.

What about global existence?

$\frac{\partial f}{\partial y} = -2y$ is not bounded (goes to ∞ as $y \to \infty$), and is not linear. So global existence is not guaranteed.