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MATH 2243: **Linear Algebra** & **Differential Equations**

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10.1: Laplace Transform Methods

Given $f(t)$ on $t \geq 0$, define the **Laplace transform** as: $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$, for all s where the improper integral converges.

Recall that in order to evaluate an improper integral $\int_0^\infty g(t)dt$, we evaluate the limit $\lim_{b\to\infty} \int_0^b g(t)dt$. If this limit exists, we say the integral converges.

Gamma Function: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is a generalization of the factorial: $n!$

$(0! = 1,\ 1! = 1,\ 2! = 1 \cdot 2,\ 3! = 3 \cdot 2 \cdot 1,\ \dots)$

For $n \in \mathbb{N}$, let $\Gamma(n+1) = n!$.

So $\Gamma(n+2) = (n+1)\Gamma(n+1) = n(n+1)\Gamma(n)$,

giving us, $\Gamma(5) = 4 \cdot \Gamma(4) = 4 \cdot 3 \cdot \Gamma(3) = 4!$, and $\Gamma(1) = 0!$, etc.

However, unlike the factorial function, Γ is continuous,

so we can also evaluate $x \in \mathbb{R}$ when $x > 0$. Particularly for fractions:

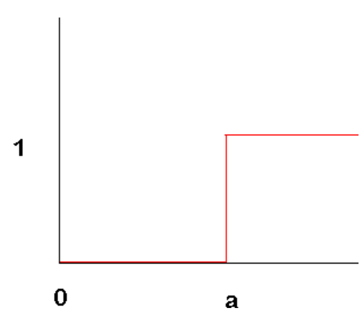
$\Gamma(\frac{1}{2}) = \sqrt{\pi}$, so $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = (\frac{3}{2} \cdot \frac{1}{2})\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$.

Linearity of Transforms: $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$.

Inverse Transforms:

Given $F(s) = \mathcal{L}\{f(t)\}$, then we call $f(t)$ the **inverse Laplace transform** of $F(s)$, and: $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Unit Step Function: $u_a(t) = \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t \geq a. \end{cases}$

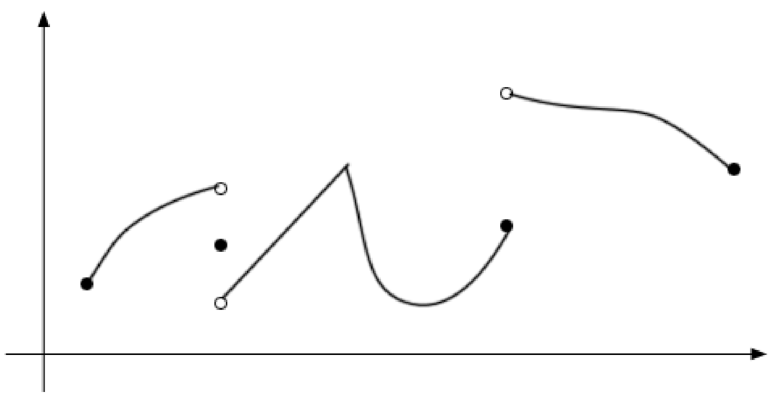


Commonly Used Transforms:

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
$1 = t^0$	$\frac{1}{s} = \frac{0!}{s^{0+1}}$	$s > 0$
t	$\frac{1}{s^2} = \frac{1!}{s^{1+1}}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^a \ (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\cos kt$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$
$\cosh kt$	$\frac{s}{s^2-k^2}$	$s > \lvert k\rvert$
$\sinh kt$	$\frac{k}{s^2-k^2}$	$s > \lvert k\rvert$
$u_0(t-a)$	$\frac{e^{-as}}{s}$	$c > 0$

where $n \in \{0, 1, 2, \ldots\}$, and $a, s, t, k \in \mathbb{R}$.

Piecewise Continuous Functions:



$f(t)$ is **piecewise continuous** on $[a,b]$ if you can carve up $[a,b]$ into a **finite** number of sub-intervals such that…

1. $f(t)$ is continuous on the interior of each of these subintervals.
2. $f(t)$ has a finite limit as t approaches each end point of each subinterval.

Furthermore, we see that $f(t)$ is piecewise continuous on $[0,\infty]$ if it is piecewise continuous on every $[0,b]$ where $b > 0$.

For a random $f(t)$, does $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ exist?

Recall that $\int_a^b f(t)dt$ exists if f is piecewise continuous on $[a,b]$.

And since e^{-st} is continuous, then $\int_0^b e^{-st} f(t)dt$ exists for all $b < \infty$ if $f(t)$ is piecewise continuous for $t \geq 0$.

Here's the hard part: Does $\int_0^b e^{-st} f(t)dt$ exists when $b \to \infty$?

It does when f is of "**exponential order**" as $t \to \infty$.

A function is of exponential order when there exists nonnegative constants M, c, and T such that $|f(t)| \leq Me^{ct}$ for $t \geq T$.

In other words, if f is eventually smaller than some exponential function. Examples where this requirement is met: all bounded functions & polynomials.

Existence and Uniqueness:

If f is piecewise continuous for $t \geq 0$, and is of exponential order as $t \to \infty$, then its Laplace transform exists. More precisely, if f is piecewise continuous and $|f(t)| \leq Me^{ct}$, then $F(s)$ exists for all $s > c$.

If f and g both have Laplace transforms (let's label them $F(s)$ and $G(s)$), and there exists some number c such that $F(s) = G(s)$ for $s > c$, then on those parts of $[0,+\infty)$ where f and g are continuous (since it is piecewise continuous), $f(t)$ is actually the same function as $g(t)$!!

If f has a Laplace transform $F(s)$, then $\lim_{s\to\infty} F(s) = 0$.

Problem: **#6** Apply the definition $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$ to find directly the Laplace transform of $f(t) = \sin^2 t$.

$$\mathcal{L}\{\sin^2 t\} = \int_0^\infty e^{-st}\sin^2 t\,dt =$$

$$= \tfrac{1}{2}\int_0^\infty e^{-st}(1 - \cos 2t)dt = \tfrac{1}{2}\int_0^\infty e^{-st}dt - \tfrac{1}{2}\int_0^\infty e^{-st}\cos 2t\,dt$$

Let's first take a look at that second integral…

$$\int_0^\infty e^{-st}\cos 2t\,dt = \left[-\tfrac{\cos 2t}{se^{st}}\right]_{t=0}^{\infty} - \tfrac{2}{s}\int_0^\infty e^{-st}\sin 2t\,dt$$

$$= \left[-\tfrac{\cos 2t}{se^{st}}\right]_{t=0}^{\infty} - \tfrac{2}{s}\left(\left[-\tfrac{\sin 2t}{se^{st}}\right]_{t=0}^{\infty} + \tfrac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt\right)$$

$$= \left[-\tfrac{\cos 2t}{se^{st}}\right]_{t=0}^{\infty} + \left[\tfrac{2\sin 2t}{s^2e^{st}}\right]_{t=0}^{\infty} - \tfrac{4}{s^2}\int_0^\infty e^{-st}\cos 2t\,dt$$

$$\left(1 + \tfrac{4}{s^2}\right)\int_0^\infty e^{-st}\cos 2t\,dt = \left[\tfrac{2\sin 2t}{s^2e^{st}} - \tfrac{\cos 2t}{se^{st}}\right]_{t=0}^{\infty} = (0-0) - \left(\tfrac{0}{s^2} - \tfrac{1}{s}\right) = \tfrac{1}{s}.$$

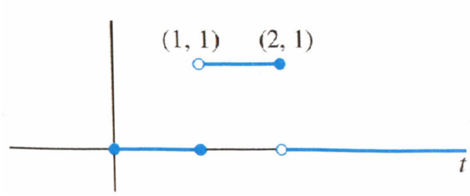
So, $$\int_0^\infty e^{-st}\cos 2t\,dt = \frac{\frac{1}{s}}{\left(1+\frac{4}{s^2}\right)} = \frac{s}{s^2+4}.$$

Recall that we were trying to solve $\mathcal{L}\{\sin^2 t\} = \tfrac{1}{2}\int_0^\infty e^{-st}dt - \tfrac{1}{2}\int_0^\infty e^{-st}\cos 2t\,dt.$

Therefore, $\mathcal{L}\{\sin^2 t\} = \frac{1}{2}[-\frac{1}{s}e^{-st}]_{t=0}^{\infty} - \frac{1}{2}\left(\frac{s}{s^2+4}\right) = \frac{1}{2}\left[(0+\frac{1}{s}) - \frac{s}{s^2+4}\right] =$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$

Problem: **#8** Apply the definition $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$ to find directly the Laplace transform of the following graph: $f(t) =$



$\mathcal{L}\{f(t)\} = \int_0^\infty [e^{-st}f(t)]dt$

$= \int_1^2 [e^{-st}\cdot 1]dt = \left[-\frac{e^{-st}}{s}\right]_1^2 = \frac{e^{-s}-e^{-2s}}{s}.$

Problem: **#14** Use the common transforms in the table above to find the transform of $f(t) = t^{\frac{3}{2}} - e^{-10t}$.

$\mathcal{L}\left\{t^{\frac{3}{2}} + e^{-10t}\right\} =$

Because of linearity:

$= \mathcal{L}\left\{t^{\frac{3}{2}}\right\} + \mathcal{L}\{e^{-10t}\}$

Using the table:

$t^a \; (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$

$= \frac{\Gamma(\frac{5}{2})}{s^{\frac{5}{2}}} + \frac{1}{s+10}$

Recall: "$\Gamma(x+2) = (x+1)\Gamma(x+1) = x(x+1)\Gamma(x)$"

So, $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2})$

$= (\frac{3}{2} \cdot \frac{1}{2})\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$.

And, $\mathcal{L}\left\{t^{\frac{3}{2}} + e^{-10t}\right\} = \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}} + \frac{1}{s+10}$.

Problem: ≈**#10. The inverse Laplace transform of the function** $\frac{9+s}{4-s^2}$ **is…**

$$\mathcal{L}^{-1}\left\{\frac{9+s}{4+s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{9}{4+s^2} + \frac{s}{4+s^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{9}{4+s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{4+s^2}\right\}$$

Using the table:

$\cos kt$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$

$$= \frac{9}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \frac{9}{2}\ \sin 2t + \ \cos 2t.$$