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8.5 Eigenvalues of Symmetric Matrices

Good news, most matrices found in application are symmetric, AND have e-vecs which form an orthogonal basis for the underlying space (complete). Let's see what else this gives us:

Theorem: Let $\mathbf{A} = \mathbf{A}^T$ be a real symmetric $n \times n$ matrix. Then,

a) All the e-vals of $\mathbf{A}$ are *real*

b) E-vecs corresponding to *distinct* e-vals are *orthogonal*.

c) There's an orthonormal basis of $\mathbb{R}^n$ consisting of n e-vecs of $\mathbf{A}$.

In particular, all real symmetric matrices are *complete* and real *diagonalizable*.

Proof quite long, in book.

Concretely: Find an orthonormal basis of $\mathbb{R}^n$ given $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, that has $\lambda_1 = 4$, $\lambda_2 = 2$ and $\vec{v}_1 = (1,1)$, $\vec{v}_2 = (-1,1)$.

Note $\mathbf{A}$ is symmetric. So, it is easily checked that $\vec{v}_1 \cdot \vec{v}_2 = 0$.

To get the orthonormal basis, we need only normalize the $\vec{v}_i$: $\vec{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\vec{u}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Theorem: A symmetric $\mathbf{K} = \mathbf{K}^T$ is positive definite **iff** all of its e-vals are strictly positive.

Proof: ($\Rightarrow$) First, if $\mathbf{K} > 0$, then, by definition, $\vec{x}^T\mathbf{K}\vec{x} > 0$ for all nonzero $\vec{x} \in \mathbb{R}^n$.

In particular, if $\vec{x} = \vec{v} \neq \vec{0}$ is an e-vec with (necessarily real) e-val λ, then

$$0 < \vec{v}^T\mathbf{K}\vec{v} = \vec{v}^T(\lambda\vec{v}) = \lambda\vec{v}^T\vec{v} = \lambda|\vec{v}|^2,$$

which immediately proves $\lambda > 0$.

Conversely ($\Leftarrow$), suppose $\mathbf{K}$ has all positive e-vals. Let $\vec{u}_1, \ldots, \vec{u}_n$ be the orthonormal e-vec basis guaranteed by the previous thm, with $\mathbf{K}\vec{u}_j = \lambda_j\vec{u}_j$ where $\lambda_j > 0$.

Writing $\vec{x} = c_1\vec{u}_1 + \ldots + c_n\vec{u}_n$, we obtain $\mathbf{K}\vec{x} = c_1\lambda_1\vec{u}_1 + \ldots + c_n\lambda_n\vec{u}_n$.

Therefore, using orthonormality of the e-vecs:

$$\vec{x}^T\mathbf{K}\vec{x} = \left(c_1\vec{u}_1^T + \ldots + c_n\vec{u}_n^T\right)(c_1\lambda_1\vec{u}_1 + \ldots + c_n\lambda_n\vec{u}_n) = \lambda_1 c_1^2 + \ldots + \lambda_n c_n^2.$$

Note this last expression is greater then zero whenever $\vec{x} \neq 0$,

since only $\vec{x} = \vec{0}$ has coordinates $c_1 = \ldots = c_n = 0$. This establishes $\mathbf{K} > \mathbf{0}$. ■

Proposition: Let $\mathbf{A} = \mathbf{A}^T$ be $n \times n$, symmetric. Let $\vec{v}_1, \ldots, \vec{v}_n$ be an orthogonal e-vec basis such that $\vec{v}_1, \ldots, \vec{v}_r$ correspond to nonzero e-vals, while $\vec{v}_{r+1}, \ldots, \vec{v}_n$ are null e-vecs corresponding to the zero e-val (if any). Then $r = rank\,\mathbf{A}$; the non-null e-vecs $\vec{v}_1, \ldots, \vec{v}_r$ form an orthogonal basis for $img\,\mathbf{A} = coimg\,\mathbf{A}$, while the null e-vecs $\vec{v}_{r+1}, \ldots, \vec{v}_n$ form an orthogonal basis for $\ker \mathbf{A} = co\ker \mathbf{A}$.

Proof: The zero e-space coincides with kernel: $V_0 = \ker(\mathbf{A} - \mathbf{0I}) = \ker \mathbf{A}$.

Thus, the linearly independent null e-vecs form a basis for $\ker \mathbf{A}$, which has dimension $n - r$ where $r = rank\,\mathbf{A}$.

Moreover, the remaining r non-null e-vecs are (by prev. thm) orthogonal to the null e-vecs. Therefore, they must

form a basis for the kernel's orthogonal complement, namely $coimg\,\mathbf{A} = img\,\mathbf{A}$. (since $\mathbf{A} = \mathbf{A}^T$) ■

Example: Determine whether $\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{bmatrix}$ is positive definite by computing its e-vals.

Validate your conclusions by using the methods learned previously in the course.

Observe the symmetry. Then: $\begin{vmatrix} 4-\lambda & -1 & -2 \\ -1 & 4-\lambda & -1 \\ -2 & -1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -1 & -2 \\ -1 & 4-\lambda & -1 \\ 0 & 2\lambda-9 & 6-\lambda \end{vmatrix}$

$$= (4-\lambda)((4-\lambda)(6-\lambda) - (9-2\lambda)) - (-1)(-(6-\lambda) - (-2)(2\lambda-9)) = -\lambda^3 + 12\lambda^2 - 42\lambda + 36.$$

Note the factors of (the constant term) 36 are $\{\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm13, \pm36\}$.

Try these as possible solutions until eventually, finding $\lambda = 6$.

Polynomial division by $(\lambda - 6)$.

$$-\lambda^3 + 12\lambda^2 - 42\lambda + 36 = (\lambda-6)(-\lambda^2) + (6\lambda^2 - 42\lambda + 36) = (\lambda-6)(-\lambda^2) + (\lambda-6)(6\lambda) + (-6\lambda + 36)$$
$$= (\lambda-6)(-\lambda^2) + (\lambda-6)(6\lambda) - 6(\lambda-6).$$

So, $-\lambda^3 + 12\lambda^2 - 42\lambda + 36 = (\lambda - 6)(-\lambda^2 + 6\lambda - 6)$,

and $\lambda \in \{6,\ 3 \pm \sqrt{3}\} \approx \{6, 1.27,\ 4.73\}$. (quadratic formula)

Pos e-vals, so pos def. Verify w/ "other method in course"?

Recall, a symmetric matrix is positive definite **iff** it is regular and has all positive pivots.

$$\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 4 & -1 & -2 \\ 0 & \frac{15}{4} & -\frac{3}{2} \\ 0 & -9 & 6 \end{bmatrix} \to \begin{bmatrix} 4 & -1 & -2 \\ 0 & \frac{15}{4} & -\frac{3}{2} \\ 0 & 0 & \frac{16}{5} \end{bmatrix}, \text{ so pos def.}$$

The Spectral Theorem

The previous section told us that a real symmetric matrix produces an e-basis for the space, and is therefore diagonalizable. Furthermore, since we can choose those e-vecs such that they are of unit length, we have the following:

Theorem: Let $\mathbf{A}$ be real, symmetric. Then there exists an orthogonal $\mathbf{Q}$ such that $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where $\mathbf{\Lambda}$ is real/diagonal. The e-vals of $\mathbf{A}$ appear on the diagonal of $\mathbf{\Lambda}$, while the columns of $\mathbf{Q}$ are the corresponding orthonormal e-vecs.

! $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ is not the same as $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^{-1}$ (the latter produced from row reduction method).

The spectral factorization provides us an alternative means of diagonalizing the associated quadratic form $q(\vec{x}) = \vec{x}^T\mathbf{A}\vec{x}$. That is, a means for completing the square.

Observe (for real symmetric $\mathbf{A}$): $q(\vec{x}) = \vec{x}^T\mathbf{A}\vec{x} = \vec{x}^T\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}\vec{x} = \vec{y}^T\mathbf{\Lambda}\vec{y} = \sum_{i=1}^n \lambda_i y_i^2$,

where $\vec{y} = \mathbf{Q}^T\vec{x} = \mathbf{Q}^{-1}\vec{x}$ are the coords of $\vec{x}$ with respect to the orthonormal e-vec basis of $\mathbf{A}$.

In particular, $q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$ and so $\mathbf{A}$ is positive definite **iff** each e-val is strictly positive,

reconfirming our previous thm.

Example: Construct a symmetric $\mathbf{A}$ with the following e-vecs and e-vals, or explain why none exists:
$\lambda_1 = -2,\ \vec{v}_1 = (1,-1)$ and $\lambda_2 = 1,\ \vec{v}_2 = (1,1)$.

It's diagonalizable (real, symmetric, so two e-vecs), and forming $\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$, we have:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$