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8.3 Eigenvector Bases

Eigenspace of $\mathbf{A}^{n\times n}$ associated with $\boldsymbol{\lambda}$:

Definition: For a particular e-val λ, the kernel of $\mathbf{A}-\lambda\mathbf{I}_n$, is denoted by $E_\lambda := \ker(\mathbf{A}-\lambda\mathbf{I}_n) = \left\{\vec{v}\in\mathbb{R}^n \;:\; \mathbf{A}\vec{v}=\lambda\vec{v}\right\}$

$= span\{\text{e-vecs assoc'd w}/\lambda\} = \{\text{all e-vecs assoc'd w}/\lambda\}\cup\{\vec{0}\}$, and is referred to as the λ **eigenspace of A**.

❗ E_λ is a subspace of $\mathbb{R}^n$ b/c kernels are subspaces.

You try: Find e-vals, e-vecs, e-spaces for: $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$. ...

$$f_{\mathbf{A}}(\lambda) = \det\begin{bmatrix} -3-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix} = (-3-\lambda)^2 = (\lambda+3)^2 \quad\Rightarrow\quad \text{E-val: } \lambda = -3.$$

$$E_{-3} = \ker(\mathbf{A}-\lambda\mathbf{I}_n) = \ker\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \ldots$$

$= \mathbb{R}^2$. (e-basis? gemu/alumu?)

e-basis: $\hat{i},\hat{j}$. $gemu(\lambda) = almu(\lambda) = 2$.

Similarly: $\mathbf{B} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$, $f_B(\lambda) = \ldots = (\lambda+3)^2$, E-val: $\lambda = -3$. ...

However, $E_{-3} = \ker(\mathbf{B}+3\mathbf{I}_n) = \ker\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} = span\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$ (a line. gemu/alumu?)

$1 = gemu(\lambda) < almu(\lambda) = 2$.

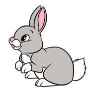
For $\mathbf{A}$: $\dim E_{-3} = 2$. For $\mathbf{B}$: $\dim E_{-3} = 1$.

Diagonalization

Motivation

$$\vec{x} = \begin{bmatrix} 2,000 \\ 1,500 \\ 10,000 \end{bmatrix}$$





In many applications (like population models), it is possible to discover a **transition matrix A**, that will transition a vector $\vec{x}$ (for example, one containing the populations of different animals in a region) from one state $\vec{x}_0$ to another state $\vec{x}_1$ (for example, from the populations in some year, to the populations in the next year). It is also used in onboard aviation software, satellite orbit maintenance, various statistics applications, and many other fields.

This is done by simply multiplying $\mathbf{A}\vec{x}_0 = \vec{x}_1$. However, we are usually interested in the long-term behavior of $\vec{x}$ (e.g., the population), so perhaps what $\vec{x}_{1000}$ will be.

But this would require us to calculate $\overset{\text{(1000 times)}}{\mathbf{AA}\ldots\mathbf{A}}\ \vec{x}_0 = \mathbf{A}^{1000}\vec{x}_0 = \vec{x}_{1000}$.

But $\mathbf{A}^{1000}$ is very difficult to calculate.

There must be a different way, right?

The Different Way

Diagonal Matrix: $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, "Zeros off of the diagonal."

So, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal!!

If we can characterize $\mathbf{A}$ as $\mathbf{A} = \mathbf{P}^{-1}\mathbf{DP}$, where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{P}, \mathbf{P}^{-1}$ are invertible matrices, then notice that:

$$\mathbf{A}^3 = (\mathbf{P}^{-1}\mathbf{DP})(\mathbf{P}^{-1}\mathbf{DP})(\mathbf{P}^{-1}\mathbf{DP}) = \mathbf{P}^{-1}\mathbf{D}(\mathbf{PP}^{-1})\mathbf{D}(\mathbf{PP}^{-1})\mathbf{DP}$$

$$= \mathbf{P}^{-1}\mathbf{D}(\mathbf{I})\mathbf{D}(\mathbf{I})\mathbf{DP} = \mathbf{P}^{-1}\mathbf{DDDP} = \mathbf{P}^{-1}\mathbf{D}^3\mathbf{P}.$$

Also note that for any diagonal matrix: $\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}^{\mathbf{n}} = \begin{bmatrix} a_1^n & 0 & \dots & 0 \\ 0 & a_2^n & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & a_n^n \end{bmatrix}.$

For example: $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}^3 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 9 & 0 \\ 0 & 25 \end{bmatrix}$

$$= \begin{bmatrix} 27 & 0 \\ 0 & 125 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & 5^3 \end{bmatrix}. \qquad \checkmark$$

As a result, to calculate $\mathbf{A}^{1000}$, we need only calculate $\mathbf{D}^{1000}$ by raising each diagonal to the 1000*th* power, and then compute $\mathbf{P}^{-1}\mathbf{D}^{1000}\mathbf{P}$ (two matrix multiplications instead of 1000).

But how can we transform $\mathbf{A}$ into $\mathbf{P}^{-1}\mathbf{D}\mathbf{P}$? This transformation is called **diagonalizing**.

Diagonalizing Criteria

Recall $\mathbf{A}^{n\times n}$ and $\mathbf{B}^{n\times n}$ are called **similar**, if there exists an invertible matrix $\mathbf{P}$, such that: $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Definition: $\mathbf{A}^{n\times n}$ is called *diagonalizable* if it is similar to a diagonal $\mathbf{D}$;
that is, if there exists diagonal $\mathbf{D}$ & invertible $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, and so $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

Criteria for Diagonalizability: $\mathbf{A}^{n\times n}$ is diagonalizable **iff** $\mathbf{A}$ has n linearly independent e-vecs $\vec{v}_i$
(note that this may be possible even if you have less than n distinct e-vals λ_k).

Proof: First we show ($\Leftarrow$), that if we have n e-vecs, then $\mathbf{A}$ is diagonalizable.

Must show existence of $\mathbf{D}, \mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

Suppose our e-vals are $\lambda_1, \dots, \lambda_n$ (perhaps not all unique) corresponding to the n e-vecs $\vec{v}_1, \dots, \vec{v}_n$,

and let $\mathbf{P} := \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}.$

Then, $\mathbf{A}\mathbf{P} = \mathbf{A}\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{A}\vec{v}_1 & \mathbf{A}\vec{v}_2 & \dots & \mathbf{A}\vec{v}_n \\ | & | & & | \end{bmatrix}$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1 \vec{v}_1 & \lambda_2 \vec{v}_2 & \dots & \lambda_n \vec{v}_n \\ | & | & & | \end{bmatrix}. \qquad \text{(by definition of eigenvector)}$$

Now consider the diagonal matrix $\mathbf{D} := \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$.

So, $\mathbf{PD} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 \vec{v}_1 & \lambda_2 \vec{v}_2 & \dots & \lambda_n \vec{v}_n \\ | & | & & | \end{bmatrix}$.

And note that above we have shown $\mathbf{AP} = \mathbf{PD}$.

And since we know that $\mathbf{P}$ is invertible (having n linearly independent column vecs), we can multiply on the right by $\mathbf{P}^{-1}$, to obtain: $\mathbf{A} = \mathbf{PDP}^{-1}$. So we have shown ($\Leftarrow$).

Next we show ($\Rightarrow$). That if $\mathbf{A}$ is diagonalizable, then we have n e-vecs.

Suppose $\mathbf{A}$ is similar to $\mathbf{D} := \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$, and let $\mathbf{P} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$

be invertible such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ or equivalently $\mathbf{AP} = \mathbf{PD}$ (this is what it means to be diagonalizable).

We must show $\vec{v}_1, \dots, \vec{v}_n$ are e-vecs & linearly independent.

We calculate, $\mathbf{AP} = \mathbf{A}\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}\vec{v}_1 & \dots & \mathbf{A}\vec{v}_n \end{bmatrix}$,

and $\mathbf{PD} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \vdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} = \begin{bmatrix} d_1 \vec{v}_1 & \dots & d_n \vec{v}_n \end{bmatrix}$.

Comparing $\mathbf{AP} = \mathbf{PD}$ component-wise, it follows that $\mathbf{A}\vec{v}_j = d_j \vec{v}_j$ for $j = 1, 2, \dots, n$.

But this defines what it is to be an e-vec/e-val. Thus $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are e-vecs of **A** associated with e-vals $d_1, d_2, \ldots, d_n$, respectively.

And it follows from previous thms that these n e-vecs **A** are linearly independent, because they are column vecs of invertible **P**. ■

So we have proven the claim that: "$\mathbf{A}^{n\times n}$ is diagonalizable **iff A** has n linearly independent e-vecs $\vec{v}_i$."

So our ability to diagonalize depends upon **A** having n linearly independent e-vecs.
The following thm is helpful in this regard:
Thm: Suppose e-vecs $\vec{v}_1, \ldots, \vec{v}_k$ are associated w/ distinct e-vals $\lambda_1, \ldots, \lambda_k$ of **A**.
Then these k e-vecs are linearly independent.

Proof: Using induction on k. Obviously true when $k = 1$, so this satisfies our base case.

Our next task is to show that if any set of $k-1$ eigenvectors associated with distinct eigenvalues is linearly independent, then any set of k eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{v}_k\}$ is also linearly independent.

In other words, that $c_1\vec{v}_1 + c_2\vec{v}_2, + \ldots + c_k\vec{v}_k = \vec{0}$, $(*)$

requires $c_1 = c_2 = \ldots = c_k = 0$. First note that $(\mathbf{A} - \lambda_j\mathbf{I})\vec{v}_j = 0$, for all j since these are e-vecs.

Observe that $(\mathbf{A} - \lambda_j\mathbf{I})\vec{v}_i = \mathbf{A}\vec{v}_i - \lambda_j\vec{v}_i = \lambda_i\vec{v}_i - \lambda_j\vec{v}_i = (\lambda_i - \lambda_j)\vec{v}_i$, for all i,j.

Therefore, if we multiply $(*)$ by $(\mathbf{A} - \lambda_1\mathbf{I})$, we get $(\lambda_2 - \lambda_1)c_2\vec{v}_2, + \ldots + (\lambda_k - \lambda_1)c_k\vec{v}_k = \vec{0}$.

Since we assumed the e-vals were distinct, and that any set of $k-1$ e-vecs is linearly independent,

this requires $c_2 = c_3 = \ldots = c_k = 0$. If we substitute these into $(*)$, the remaining eqn is:

$c_1\vec{v}_1 = 0$, but since we know that e-vecs are nontrivial, it must be that $c_1 = 0$. Having shown that all $c_i = 0$,

we can conclude that the k e-vecs are linearly independent, and by induction the theorem follows. ■

Conclusion: So if we find n such distinct e-vals, our matrix is diagonalizable.

However, this does **NOT** mean that if you find $k < n$ distinct e-vals that your matrix is undiagonalizable.
Rather, it means you must calculate your e-vecs to see if your k e-vals *nonetheless* generate n e-vecs.

The following theorem is a natural consequence:

Theorem: If real $\mathbf{A}^{n\times n}$ has n distinct **real** e-vals $\lambda_1, \ldots, \lambda_n$, then the corresponding real e-vecs $\vec{v}_1, \ldots, \vec{v}_n$ form a basis of $\mathbb{R}^n$.

If **A** (which may now be either real or complex) has n distinct **complex** e-vals, then the corresponding e-vecs $\vec{v}_1, \ldots, \vec{v}_n$ form a basis of $\mathbb{C}^n$.

Definition: An e-val λ of **A** is called *complete* if $gemu(\lambda) = almu(\lambda)$. **A** is said to be *complete* if all the e-vals are complete.

Diagonalizing Algorithm

Upon calculating the e-vals and e-vecs of **A**, if you find n linearly independent e-vecs, then you can construct eqn $\mathbf{A} = \mathbf{PDP}^{-1}$ where **P** is invertible, and **D** diagonal.

♦ Arrange e-vals along principal diagonal of an otherwise zero matrix (including non-distinct λ_ks if any).

For example, $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

♦ Arrange corresponding e-vecs vertically as columns in a new matrix

(in same order as you did the λ_ks): $\mathbf{P} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix}$.

♦ Calculate the inverse $\mathbf{P}^{-1}$. Now you have: $\mathbf{A} = \mathbf{PDP}^{-1}$, done!

Theorem: A matrix is complex diagonalizable **iff** it is complete.

(So $\mathbf{D} = \mathbf{S}^{-1}\mathbf{AS}$, where cols of **S** are e-basis)

A real matrix is real diagonalizable **iff** it is complete and has all real eigenvalues.

Pop-Quiz: If $almu(\lambda) = 2$, $gemu(\lambda) =$?? ...

1 or 2.

Example: Suppose $f_{\mathbf{A}}(\lambda) = -(\lambda-7)^2(\lambda-8)$ and **A** diagonalizable (almu/gemu?) ...

⇔ **A** has e-basis (so, 3 indep. e-vecs)

$\Leftrightarrow \quad gemu(7) = 2.$

Example: Suppose $f_{\mathbf{A}}(\lambda) = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$. Why is **A** diagonalizable? ...

$gemu(\lambda)$ is always at least one, so I get 3 different e-vecs, which form an e-basis.

Exercises

Problem: **Determine whether** $\mathbf{A} = \begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$ **is diagonalizable.**

If it is, find a diagonalizing matrix P and a diagonal matrix D such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

Must use: $|\mathbf{A} - \lambda \mathbf{I}| = 0$, $\begin{vmatrix} 6-\lambda & -5 & 2 \\ 4 & -3-\lambda & 2 \\ 2 & -2 & 3-\lambda \end{vmatrix} = 0$

$$= \begin{vmatrix} 1-\lambda & -5 & 2 \\ 1-\lambda & -3-\lambda & 2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -5 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda).$$

Eigenvalues: $\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$

Recall: if an $n \times n$ matrix **A** has n distinct eigenvalues, then it is diagonalizable!
(this is because $\lambda_1, \lambda_2, \lambda_3$ will necessarily produce three linearly independent eigenvectors.)

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Must use: $(\mathbf{A} - \lambda_k \mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_1 = 1$: $\begin{bmatrix} 6-1 & -5 & 2 \\ 4 & -3-1 & 2 \\ 2 & -2 & 3-1 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 2 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow z = 0, \ y = b, \ x = b$

$$\vec{v}_1 = \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ when } b = 1.$$

With $\lambda_2 = 2$: $\begin{bmatrix} 4 & -5 & 2 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} \Rightarrow z = c,\ y = 0,\ x = -\frac{1}{2}c$

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2}c \\ 0 \\ c \end{bmatrix} = \frac{1}{2}c\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \text{ when } c = 2.$$

With $\lambda_3 = 3$: $\begin{bmatrix} 3 & -5 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 6 & -6 \\ 0 & 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \Rightarrow z = c, \qquad y = c, \qquad x = c.$$

$$\vec{v}_3 = \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ when } c = 1.$$

So, $\mathbf{S} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$.

And, $\mathbf{A} = \mathbf{SDS^{-1}}$, where $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Problem: **Determine whether or not the given matrix A is diagonalizable. If it is, find a diagonalizing matrix S, and a diagonal matrix D such that $\mathbf{S^{-1}AS = D}$.**

$$\mathbf{A} = \begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Must use: $|\mathbf{A} - \lambda\mathbf{I}| = 0$. Which is: $\begin{vmatrix} -2-\lambda & 4 & -1 \\ -3 & 5-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix}$

$$= \begin{vmatrix} 2-\lambda & 4 & -1 \\ 2-\lambda & 5-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 4 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda)^2.$$

Eigenvalues: $\lambda_{1,2} = 1,\ \lambda_3 = 2.$ (diagonalizable?)

Must use: $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_{1,2} = 1$: $\left.\begin{aligned} -3a + 4b - c &= 0 \\ -3a + 4b - c &= 0 \\ -a + b &= 0 \end{aligned}\right\} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, (uh oh! only one.)

With $\lambda_3 = 2$: $\left.\begin{aligned} -4a + 4b - c &= 0 \\ -3a + 3b - c &= 0 \\ -a + b - c &= 0 \end{aligned}\right\} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(leave it to you to verify above as an exercise)

The given matrix **A** has only the two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$, and therefore is not diagonalizable.

Problem: Let **A** be a 3×3 matrix with three distinct eigenvalues.

Tell how to construct six different invertible matrices $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_6$

and six different diagonal matrices $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_6$ such that $\mathbf{P}_i\mathbf{D}_i(\mathbf{P}_i)^{-1} = \mathbf{A}$ for each $i = 1, 2, \ldots, 6$.

Three eigenvectors associated with three distinct eigenvalues can be arranged with six different permutations as the column vectors of the invertible matrix: $\mathbf{P} = \begin{bmatrix} \vec{v}_i & \vec{v}_j & \vec{v}_k \end{bmatrix}$.

$\mathbf{P}_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$, $\mathbf{P}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_3 & \vec{v}_2 \end{bmatrix}$, $\mathbf{P}_3 = \begin{bmatrix} \vec{v}_2 & \vec{v}_1 & \vec{v}_3 \end{bmatrix}$,

$\mathbf{P}_4 = \begin{bmatrix} \vec{v}_2 & \vec{v}_3 & \vec{v}_1 \end{bmatrix}$, $\mathbf{P}_5 = \begin{bmatrix} \vec{v}_3 & \vec{v}_2 & \vec{v}_1 \end{bmatrix}$, $\mathbf{P}_6 = \begin{bmatrix} \vec{v}_3 & \vec{v}_1 & \vec{v}_2 \end{bmatrix}$.