Applied Linear Algebra

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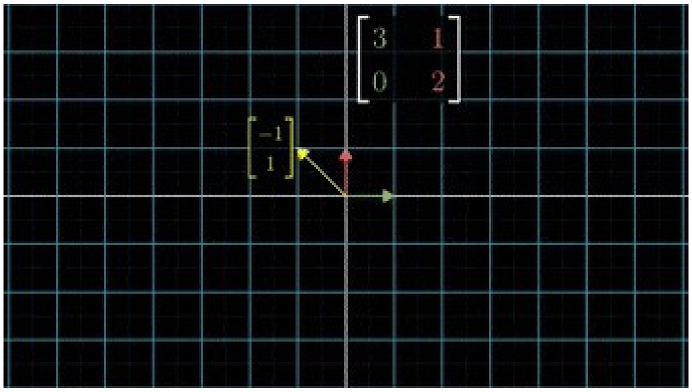
8.2 Eigenvalues and Eigenvectors

Eigenvalue/Eigenvector Intuition:



[see animation in class]

Definition: Given $\mathbf{A}^{n\times n}$, scalar λ is called *eigenvalue* of $\mathbf{A}$ if there's $\vec{v} \neq 0$, called an *eigenvector*, such that: $\mathbf{A}\vec{v} = \lambda\vec{v}$. $(*)$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ 0 & 2 & 1 \end{bmatrix}$. Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

If $\mathbf{A}\vec{v} = \lambda\vec{v}$, then $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\Rightarrow \quad \begin{aligned} x + 2y \quad &= \lambda x, \\ 4x \quad + 6z &= \lambda y, \\ 2y + z &= \lambda z. \end{aligned} \quad \Rightarrow \quad \text{4 vars, 3 eqns, nonlinear! (ick!)}$$

There must be a different way, right?

The Different Way

For some $\mathbf{A}$, assume λ exists. Recall (by definition of e-val) that for every λ there exists a nonzero $\vec{v} \in \mathbb{R}^n$ such that

$\mathbf{A}\vec{v} = \lambda\vec{v}$ ...

$\Leftrightarrow \quad \mathbf{A}\vec{v} - \lambda\vec{v} = \vec{0} \quad \Rightarrow \quad \mathbf{A}\vec{v} - \lambda\mathbf{I}_n\vec{v} = \vec{0}$

$\Leftrightarrow \quad (\mathbf{A} - \lambda\mathbf{I}_n)\vec{v} = \vec{0}$ ...

$\Leftrightarrow$ Therefore: $\ker(\mathbf{A} - \lambda\mathbf{I}_n) \neq \{\vec{0}\}$

$\Leftrightarrow$ $\mathbf{A} - \lambda\mathbf{I}_n$ cannot be invertible (i.e., must be singular)

$\Leftrightarrow$ $\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}_n) = 0.$

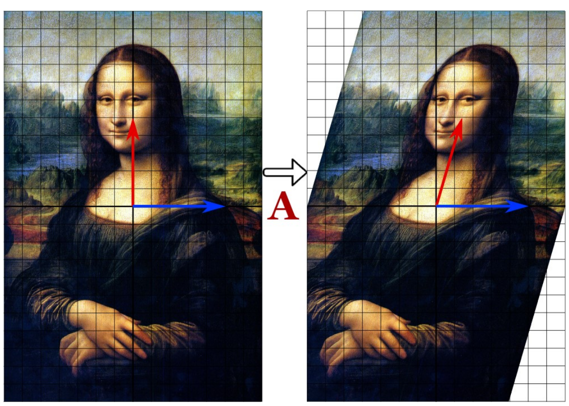
We could solve this! (...if polynomial in λ is of low degree)

We've just shown the following results:

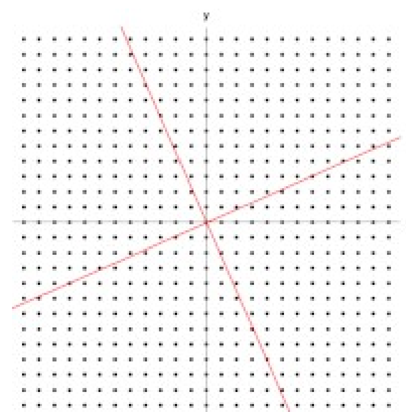
Theorem: A scalar λ is an e-val of $\mathbf{A}^{n\times n}$ **iff** $\mathbf{A} - \lambda\mathbf{I}$ is singular, i.e., of rank$< n$.

The corresponding e-vecs are the nonzero solutions to the *eigenvalue equation*: $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$.

Corollary: A scalar λ is an e-val of **A** **iff** λ is a solution to *characteristic equation/polynomial*: $f_{\mathbf{A}}(\lambda) := |\mathbf{A} - \lambda\mathbf{I}| = 0$.



vector (red), eigenvector (blue) under **A**



(see animated during class)

Corollary: A matrix $\mathbf{A}^{n\times n}$ is singular **iff** **A** has e-val: $\lambda = 0$.

Proof: $\Leftarrow$: $|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{A} - 0\mathbf{I}| = |\mathbf{A}| = 0.$

$\Rightarrow$ **A** is singular implies , $\vec{v} \neq \vec{0}$ such that $\mathbf{A}\vec{v} = \vec{0} = \lambda\vec{v}$, where $\lambda = 0$.

Example: Find e-vals of $\mathbf{A} = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 4 & 7 \\ 0 & 0 & 7 \end{bmatrix}$. ...

$$\det(\mathbf{A}-\lambda\mathbf{I}_3) = 0 \quad \Rightarrow \quad \det\begin{bmatrix} 5-\lambda & 1 & 2 \\ 0 & 4-\lambda & 7 \\ 0 & 0 & 7-\lambda \end{bmatrix} = 0 \qquad \ldots$$

$$\Rightarrow \quad (5-\lambda)(4-\lambda)(7-\lambda) = 0 \quad \Rightarrow \quad \lambda \in \{4,5,7\}.$$ So, as we can see:

E-vals of Triangular Matrix Thm: E-vals of a triangular matrix are its diagonal entries.

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ 0 & 2 & 1 \end{bmatrix}$, find e-vals.

$$f_{\mathbf{A}}(\lambda) = \det\left(\mathbf{A} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 4 & -\lambda & 6 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -\lambda & 6 \\ 2 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 4 & -\lambda \\ 0 & 2 \end{vmatrix}$$

$$= (1-\lambda)[-\lambda(1-\lambda)-12] - 2[4(1-\lambda)] \qquad \ldots$$

$$= (1-\lambda)(-\lambda(1-\lambda)-12-8) = (1-\lambda)(\lambda^2-\lambda-20)$$ (Pro-tip, keep common factor, cubics are hard!)

$$= (\lambda-1)(\lambda-5)(\lambda+4).$$

e-vals of $\mathbf{A}$ are $1, -4, 5$.

How do we find e-vecs of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ 0 & 2 & 1 \end{bmatrix}$?

When $\lambda_1 = -4$: Solve $(\mathbf{A}-\lambda\mathbf{I})\vec{v} = \vec{0}$ or find $\ker(\mathbf{A}-\lambda\mathbf{I})$. (3 eqs, 3 vars!)

$$\Rightarrow \begin{bmatrix} 1+4 & 2 & 0 \\ 4 & 0+4 & 6 \\ 0 & 2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 0 \\ 4 & 4 & 6 \\ 0 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -6 \\ 4 & 4 & 6 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -6 \\ 0 & 12 & 30 \\ 0 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -6 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker \mathbf{A} = \left\{ \begin{bmatrix} z \\ -\frac{5}{2}z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix} \right\}.$$

So $\vec{v}_1 = \langle 2 \;\; -5 \;\; 2 \rangle$ is e-vec for $\lambda_1 = -4$.

? **Sometimes.**

Our e-vec from above $\vec{v}_1 = \langle 2 \;\; -5 \;\; 2 \rangle$ implies that $2\vec{c}_1 - 5\vec{c}_2 + 2\vec{c}_3 = \vec{0}$, where $\vec{c}_i$ are column vecs of $\mathbf{A} - \lambda\mathbf{I}$.

If you are sufficiently fancy, you may be able to observe this directly from $\mathbf{A} - \lambda\mathbf{I}$, without the above calculations.

$$\begin{matrix} 2 & -5 & 2 \end{matrix}$$

$$\mathbf{A{-}\lambda I} = \begin{bmatrix} 5 & 2 & 0 \\ 4 & 4 & 6 \\ 0 & 2 & 5 \end{bmatrix}$$

The numbers placed above $\mathbf{A} - \lambda\mathbf{I}$, while attempting this process, are called **Kyle numbers**.

❗ With Kyle # method, you've only determined $\vec{v}_1 \in \ker(\mathbf{A} - \lambda\mathbf{I})$,

so the kernel is at least as big as $span\{\vec{v}_1\}$. Could it be larger?

$$\lambda_2 = 1 : \ker(\mathbf{A} - \mathbf{I}) = \ker \begin{bmatrix} 0 & 2 & 0 \\ 4 & -1 & 6 \\ 0 & 2 & 0 \end{bmatrix} \qquad \ldots$$

$$\begin{matrix} 3 & 0 & -2 \end{matrix}$$

$$= \ker \begin{bmatrix} 0 & 2 & 0 \\ 4 & -1 & 6 \\ 0 & 2 & 0 \end{bmatrix} \;(= \text{or} \supset \;?)\; span \left\{ \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

$\lambda_3 = 5 : \ker(\mathbf{A} - 5\mathbf{I}) = \ker\begin{bmatrix} -4 & 2 & 0 \\ 4 & -5 & 6 \\ 0 & 2 & -4 \end{bmatrix}$ ...

1 2 1

$= \ker\begin{bmatrix} -4 & 2 & 0 \\ 4 & -5 & 6 \\ 0 & 2 & -4 \end{bmatrix}$ (= or $\supset$?) $span\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$.

Recall: $\lambda_1 = -4 : \ker(\mathbf{A} + 4\mathbf{I}) = span\left\{\begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix}\right\}$.

Algebraic Multiplicity of λ ($almu(\lambda)$): Root multiplicity of $f_{\mathbf{A}}(\lambda)$.

Geometric Multiplicity of λ ($gemu(\lambda)$): $\dim(\ker(\mathbf{A} - \lambda\mathbf{I}_n))$.

? – Later: we find out the (= or $\supset$?) above should be equal signs because $1 \le gemu(\lambda) \le almu(\lambda)$, and every *almu* here is 1, so *gemu* is 1 too. So, once we've found 1 dim worth of e-vecs, we're done.

Eigen-stuff Gets Complex

Remark: If $a + ib$ is an e-val of **real** matrix $\mathbf{A}^{n \times n}$, w/associated e-vec $\vec{u} + i\vec{w}$, then $a - ib$ is *also* an e-val of **A**, w/e-vec $\vec{u} - i\vec{w}$.

Proof: By definition, $\mathbf{A}(\vec{u} + i\vec{w}) = (a + ib)(\vec{u} + i\vec{w})$.

Taking conjugate of both sides: $\overline{\mathbf{A}(\vec{u} + i\vec{w})} = \overline{(a + ib)(\vec{u} + i\vec{w})}$.

Recall, to take a conjugate of a vect. or matrix is to take the conjugate of each component.

So, a real matrix is unaffected by complex conjugation, $\overline{\mathbf{A}} = \mathbf{A}$, we conclude

$\Rightarrow \quad \overline{\mathbf{A}}\,\overline{(\vec{u} + i\vec{w})} = \mathbf{A}(\vec{u} - i\vec{w}) = (a - ib)(\vec{u} - i\vec{w}).$ $\qquad$ ($\overline{\mathbf{A}} = \mathbf{A}$ since $\mathbf{A} \in \mathbb{R}^{n \times n}$)

So $a - ib$ is an e-val of **A**, with associated e-vec $\vec{u} - i\vec{w}$. ■

Example: Find the e-vals & *e-spaces* (subspaces spanned by e-vecs) for $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. ...

$$\det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i. \qquad \ldots$$

$$\lambda_+ : \quad \ker\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \qquad \text{(Kyle?)} \qquad \ldots$$

$$= span\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

$$\lambda_- : \quad \ker\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \qquad \ldots$$

$$= span\begin{bmatrix} -i \\ 1 \end{bmatrix}. \qquad \ldots$$

Observe that $\vec{u}_1 = \langle i, 1\rangle$ and $\vec{u}_2 = \langle -i, 1\rangle$ are complex conjugates:

$$\overline{\vec{u}_1} = \overline{\langle i, 1\rangle} = \left\langle \overline{i}, \overline{1} \right\rangle = \langle -i, 1\rangle = \vec{u}_2.$$

Other Cool & Useful Odds & Ends

For a generic: $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\qquad f_{\mathbf{A}}(\lambda) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$$= (a-\lambda)(d-\lambda) - bc \quad = \quad \lambda^2 - (a+d)\lambda + (ad-bc) \qquad \ldots$$

$= \lambda^2 - tr(\mathbf{A})\lambda + \det\mathbf{A}$, where $tr(\mathbf{A})$ is called the *trace of* **A**, the sum of the diagonal elements.

$\Rightarrow$ e-vals of *ANY* $\mathbf{A}^{2\times 2}$ are: $\lambda = \frac{tr(\mathbf{A}) \pm \sqrt{(tr\mathbf{A})^2 - 4\det\mathbf{A}}}{2}$.

Proposition: In general, $f_{\mathbf{A}}(\lambda) = (-\lambda)^n + (tr\mathbf{A})(-\lambda)^{n-1} + \ldots + \det\mathbf{A}$.

Observe that $f_{\mathbf{A}}(0) = \det(\mathbf{A} - 0I) = \det\mathbf{A}$.

According to the **fundamental theorem of algebra**, every complex polynomial of degree $n \geq 1$ can be *completely* factored, and so we can write the characteristic polynomial as: $f_{\mathbf{A}}(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$.

The λ_i are the roots of $f_{\mathbf{A}}(\lambda)$, and hence the eigenvalues of **A**.

Corollary: Any $\mathbf{A}^{n \times n}$ possesses at *least* one and at *most* n distinct complex e-vals.

Proposition: With n real e-vals, including multiplicity:

$tr\mathbf{A} = \lambda_1 + \ldots + \lambda_n, \qquad \det\mathbf{A} = \lambda_1 \ldots \lambda_n.$

❗ This can be a timesaver, especially for 2×2 matrices:

Example: $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.

Obviously $tr\mathbf{A} = 5$ and $\det\mathbf{A} = 4$. Thms above say:

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= 5 \\ \lambda_1\lambda_2 &= 4 \end{aligned}\right\}$$

In other words, which two numbers sum to 5, and multiply to 4?

$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 4.$

Notice: The first equation gives us $\lambda_1 = 5 - \lambda_2$. Substituting into 2nd Eq: $(5 - \lambda_2)\lambda_2 - 4 = 0$.

This is $\lambda^2 - 5\lambda + 4$, the characteristic polynomial.

But earlier we didn't have to write the polynomial out. Wahoo!

Example: Given $\mathbf{A}^{3 \times 3}$ such that $tr(\mathbf{A}) = -3$ and $\det(\mathbf{A}) = -5$. Let $\vec{v} \in \mathbb{R}^3$ such that $\mathbf{A}\vec{v} = 2\vec{v}$.

What are e-vals of **A** and their multiplicities?

$-3 = 2 + \lambda_2 + \lambda_3, \qquad -5 = 2\lambda_2\lambda_3.$ (2 eqs, 2 vars!)

Proposition: Square matrices **A** and $\mathbf{A}^T$ have same characteristic eqn, and hence same e-vals with same multiplicities (but possbly different e-vecs).

Proof: This follows immediately from fact that $|\mathbf{A}^T| = |\mathbf{A}|$, learned earlier.

Observe: $f_{\mathbf{A}}(\lambda) = |\mathbf{A} - \lambda\mathbf{I}|$

$$= \left|(\mathbf{A} - \lambda \mathbf{I})^T\right|$$

$$= |\mathbf{A}^T - \lambda \mathbf{I}| = f_{\mathbf{A}^T}(\lambda). \qquad \blacksquare$$

Video Tutorial (visually rich and intuitive): https://youtu.be/PFDu9oVAE-g

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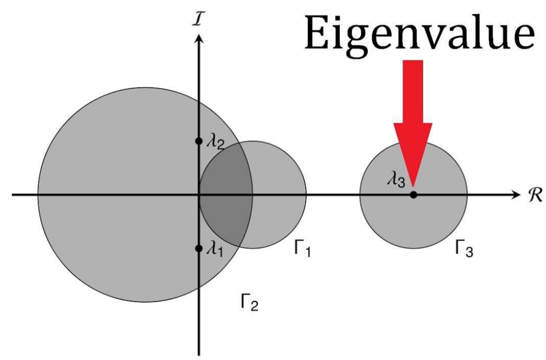
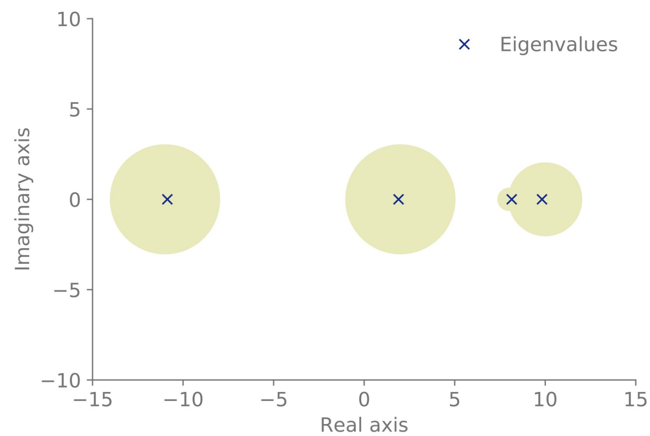
The Gershgorin Circle Thm



Definitions: Given $\mathbf{A}^{n \times n} = [a_{ij}]$, either real or complex. For each $1 \le i \le n$, define the i^{th} *Gershgorin Disk* as: $D_i = \{z \in \mathbb{C} : |z - a_{ii}| \le r_i\},$.
where $r_i = \sum_{j=1, j \ne i}^{n} |a_{ij}|$ (abs sum of ith row's components, except diagn.).
The *Gershgorin Domain* $D_{\mathbf{A}} = \cup_{i=1}^{n} D_i \subset \mathbb{C}$ is union of Gershgorin disks.

Thm: All real and complex e-vals of $\mathbf{A}$ lie in its Gershgorin domain $D_{\mathbf{A}} \subset \mathbb{C}$.

Concretely: Let $\mathbf{A} = \begin{bmatrix} 10 & 1 & 0 & 1 \\ \frac{1}{5} & 8 & \frac{1}{5} & \frac{1}{5} \\ 1 & 1 & 2 & 1 \\ -1 & -1 & -1 & -11 \end{bmatrix}$



For each row, we add up the absolute values of the non-diagonal entries.

These become the radii around each of diagonal entries (shaded yellow).
$D(10, 2),\ D(8, \frac{3}{5}),\ D(2, 3)$, and $D(-11, 3)$.

The actual eigenvalues are marked as × in the graph, and are:
$\approx \{10,\ 7.9,\ 1.9,\ -10.9\}$.

Definition: A square matrix $\mathbf{A}$ is called *strictly diagonally dominant* if $|a_{ii}| > \sum_{j=1, j \ne i}^{n} |a_{ij}| = r_i$, for all $i = 1, \ldots, n$. $\qquad (**)$

Theorem: A strictly diagonally dominant matrix is nonsingular.

Proof: The diagonal dominance inequalities $(**)$ imply radius of the ith Gershgorin disk is strictly less than modulus of its center: $r_i < |a_{ii}|$.

This implies that the disk cannot contain 0.

Indeed, if $z \in D_i$, then, by the reverse triangle inequality ($|x - y| \geq ||x| - |y||$),

$r_i > |a_{ii} - \lambda| \geq |a_{ii}| - |\lambda| > r_i - |\lambda|$, and hence $|\lambda| > 0$.

Thus, 0 does not lie in the Gershgorin domain $D_{\mathbf{A}}$, and so cannot be an e-val.

Therefore, from previous corollary above, **A** cannot be singular.

(**A** is singular implies , $\vec{v} \neq \vec{0}$ such that $\mathbf{A}\vec{v} = \vec{0} = \lambda\vec{v}$, where $\lambda = 0$.) ■

Exercises 

Problem: Find the (real) eigenvalues, the associated eigenvectors, and a basis for each eigenspace for:

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)((4-\lambda)(-1-\lambda)+6) \qquad \text{(pro tip....)}$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2) \qquad = \qquad -(\lambda-1)(\lambda-2)^2.$$

Characteristic Polynomial: $p(\lambda) = -(\lambda-1)(\lambda-2)^2 = 0.$

Eigenvalues: $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = 2.$ Now what?

For each λ_k, solve $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_1 = 1$: $\begin{bmatrix} 4-1 & -3 & 1 \\ 2 & -1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1+(-1)R_2}{\Rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_2+(-1)R_1}{\Rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad z = 0,\ y = b,\ x = y = b.$$

$$\Rightarrow \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ when } b = 1.$$

The eigenspace of $\lambda_1 = 1$ is 1-dimensional.
Basis for λ_1 eigenspace: $\{\vec{v}_1\}$.

With $\lambda_{2,3} = 2$: $\qquad \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 4-2 & -3 & 1 \\ 2 & -1-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix}$

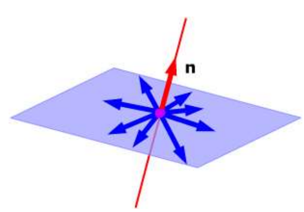
$$= \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, \quad z = c, \quad y = b, \quad x = \tfrac{3}{2}y - \tfrac{1}{2}z = \tfrac{3}{2}b - \tfrac{1}{2}c.$$

$$\Rightarrow \begin{bmatrix} \frac{3}{2}b - \frac{1}{2}c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \text{ and } \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \text{ when } b, c = 2.$$

The eigenspace of $\lambda_{2,3} = 2$ is two-dimensional.
Basis for $\lambda_{2,3}$ eigenspace: $\{\vec{v}_2, \vec{v}_3\}$.



Problem: Find the complex-conjugate eigenvalues and corresponding eigenvectors of the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix}.$$

Characteristic polynomial: $p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 0-\lambda & -12 \\ 12 & 0-\lambda \end{vmatrix}$

$= \lambda^2 + 144 = 0.$

Eigenvalues: $\lambda_1 = -12i,\ \lambda_2 = +12i.$

For each λ_k, solve $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_1 = -12i$: $\begin{bmatrix} 0-\lambda_1 & -12 \\ 12 & 0-\lambda_1 \end{bmatrix} = \begin{bmatrix} 12i & -12 \\ 12 & 12i \end{bmatrix}$

$\overset{\frac{1}{12}R_{1,2}}{\Rightarrow} \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Rightarrow} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow\ y = b$ and $x = -ib$.

So, $\vec{v}_1 = \begin{bmatrix} -ib \\ b \end{bmatrix} = b\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, when $b = 1$.

Similarly…

With $\lambda_2 = +12i$: $\left.\begin{matrix} -12ia - 12b = 0 \\ 12a - 12ib = 0 \end{matrix}\right\}$ $\quad \vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

(leave it to you as an exercise)

Note that $\vec{v}_1$ and $\vec{v}_2$ are conjugate to each other.

Problem: **Give an example of a 2 × 2 matrix A such that A and $\mathbf{A}^T$ do not have the same eigenvectors**.

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ with characteristic equation $(\lambda - 1)^2 = 0$ and the single eigenvalue $\lambda = 1$.

Then $\mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and it follows that the only associated eigenvector is a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The transpose $\mathbf{A}^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the same characteristic equation and eigenvalue,

but $\mathbf{A}^T - \mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so its only eigenvector is a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalue but different eigenvectors.