Applied Linear Algebra

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7.2 Linear Transformations

Consider linear function $L : \mathbb{R}^n \to \mathbb{R}^n$ that maps nD Euclidean space to itself. L maps a $\vec{x} \in \mathbb{R}^n$ to $L[\vec{x}] = \mathbf{A}^{n \times m}\vec{x}$.

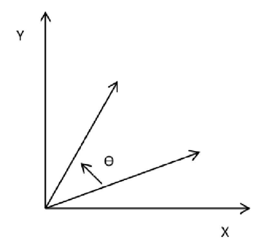
When thinking of it geometrically like this, we refer to L as a linear transformations (LT).

In $\mathbb{R}^2$, they have the form: $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$,

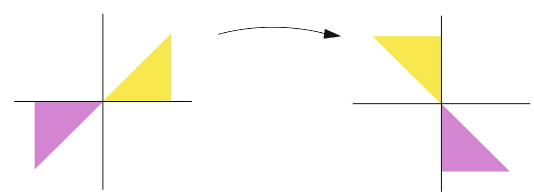
where $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Types of LTs in $\mathbb{R}^2$

Rotation Matrices: $\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, rotating vectors through angle θ.

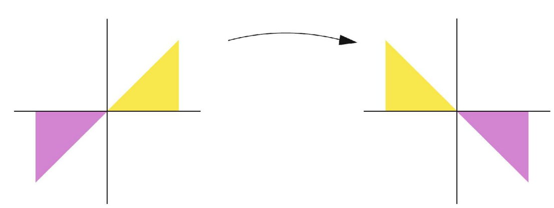


Rotations **without reflections** occur with proper orthogonal matrices.
That is, matrices $\mathbf{Q}$ that satisfy: $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ *and* $\det \mathbf{Q} = +1$.



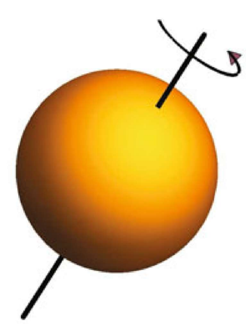
Rotations without reflections

Alternatively, rotations **with reflections** occur with improper orthogonal matrices:
$\mathbf{Q}$ Such that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $|\mathbf{Q}| = -1$.



Rotations with reflections

A proper 3D orthogonal matrix $\mathbf{Q} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$ with $\vec{u}_i = \mathbf{Q}\hat{e}_i$ corresponds to rotation in which $\hat{e}_i$ are rotated to new positions given by orthonormal basis $\vec{u}_i$. Every 3×3 orthogonal matrix corresponds to rotation around a line through the origin in $\mathbb{R}^3$ (axis of rotation).



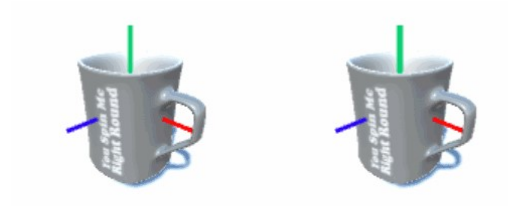
Since the product of two (proper) orthogonal matrices is also (proper) orthogonal, the composition of two rotations is also a rotation.

Unlike 2D case, order in which rotations are performed in 3D is important.

Multiplication of $n \times n$ orthogonal matrices is not commutative when $n \geq 3$.

Example: Rotating first around y-axis, then around x-axis, and then x-axis.

Not the same as around x-axis, then y-axis, then x-axis.



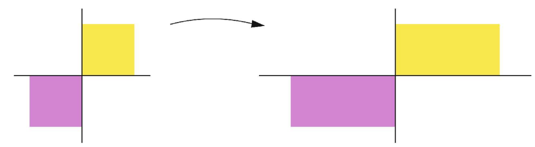
$\frac{\pi}{2}$ around y, then x, then x. $\frac{\pi}{2}$ around x, then y, then x.

see animation in class

LTs from elementary matrices:

- Multiplying a row by a scalar $\Rightarrow$ stretching and/or reflection transformation.

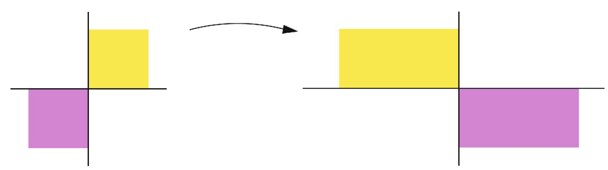
Example: $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}.$



stretching

Example: $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$

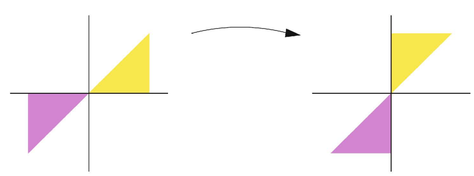
corresponds to reflection through y-axis followed by previous stretching.



reflection through y-axis then stretching

- Row interchange $\Rightarrow$ reflection through diagonal $y = x$.

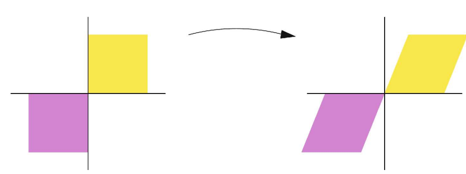
Example: $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$



reflection through $y = x$

- Adding a multiple of one row to another $\Rightarrow$ Shear along some axis.

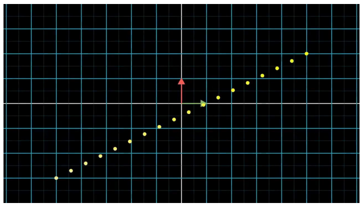
Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ y \end{bmatrix}.$



Shear

Noninvertible LTs $\Rightarrow$ projection from the plane onto the origin, or onto a line through the origin.

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 0 \end{bmatrix}$.



See animation in class

Generalizing

In higher dimensions, elementary matrices can produce LTs of types:

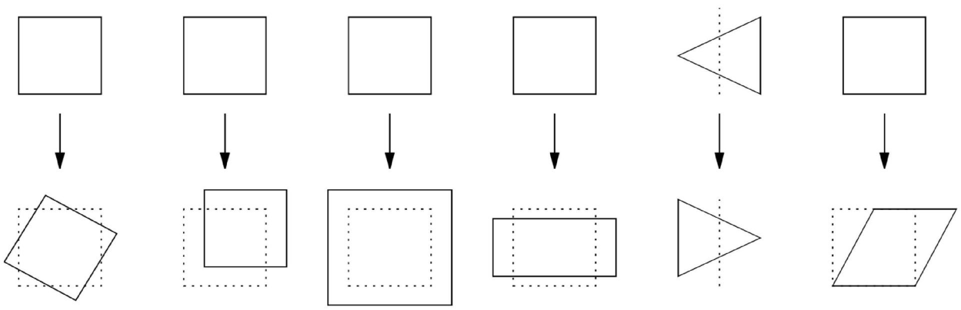
- **stretch** in a single coordinate direction.
- **reflect** through a coordinate (hyper)plane.
- **reflect** through a diagonal (hyper)plane.
- **shear** along a coordinate axis.

And *all* invertible LTs can be constructed from a sequence of these types.

The last type of LT is noninvertible, and so is not the result of an elementary matrix:

- orthogonally **project** onto a lower dimensional subspace.

And *ALL* transformations can be built up from the previous five types.



Change of Basis

Up until now, vector notation $\langle x_1, x_2, x_3 \rangle$ was meant to indicate: $x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$.

Indeed, when you have seen matrix multiplication $\mathbf{A}\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\vec{x}$, we have intended it to mean:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}(x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3) \qquad \text{(Std unit basis } \hat{e}_1, \hat{e}_2, \hat{e}_3 \text{ in domain)}$$

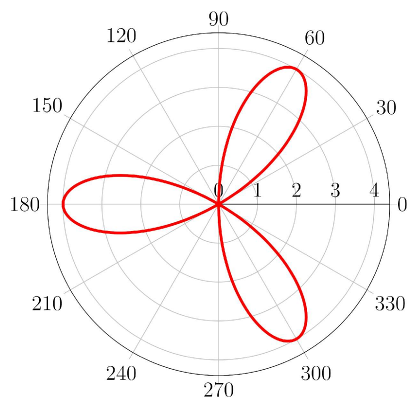
$$= x_1 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \widehat{e}_1 + x_2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \widehat{e}_2 + x_3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \widehat{e}_3$$

$$= x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= (x_1 + 2x_2 + 3x_3)\widehat{e}\,'_1 + (4x_1 + 5x_2 + 6x_3)\widehat{e}\,'_2. \qquad \text{(Std unit basis } \widehat{e}\,'_1, \widehat{e}\,'_2 \text{ in codomain)}$$

However, observe that this relies on the choice of standard unit basis vectors.

In some applications, one can gain additional insight, or create computational efficiency by adopting a different basis.

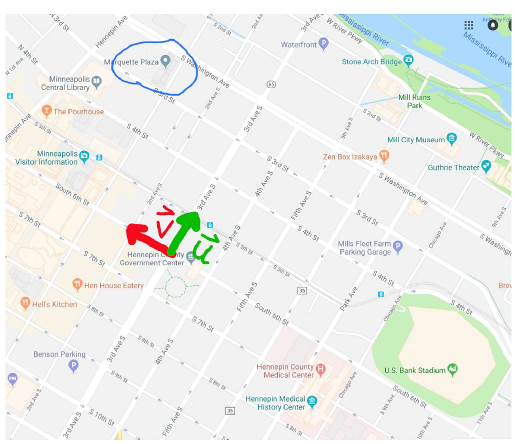


$$r = 4\cos(3\theta)$$

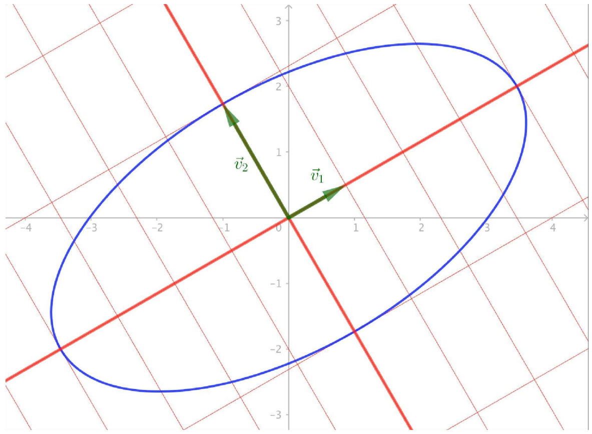
Polar coordinates make this shape simple to express.

In cartesian coordiates, we have: $x^4 + y^4 + 2x^2y^2 - 4x^3 + 12xy^2 = 0.$ (!?!)

Similarly, in linear algebra, coordinates can make all the difference.



$\vec{u}, \vec{v}$-coordinates better than $\widehat{e}_1, \widehat{e}_2$



Geometry may suggest convenient coordinates

Definition: If $\vec{x} = c_1\vec{v}_1 + \ldots + c_m\vec{v}_m$, where $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_m\}$, then we say $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$ are the **$\mathcal{B}$-coordinates of $\vec{x}$**.

❗ $\vec{x}$ in *standard coordinates* is denoted simply as $\vec{x}$.

$\vec{x}$ in $\mathcal{B}$ *−coordinates* is denoted $[\vec{x}]_{\mathcal{B}}$. …

Theorem: Let $L : V \to W$ be a linear function. Suppose V has basis $\mathcal{B}_v = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and W has basis $\mathcal{B}_w = \{\vec{w}_1, \ldots, \vec{w}_m\}$. We can write: $\vec{v} = x_1\vec{v}_1 + \ldots + x_n\vec{v}_n \in V$, $\vec{w} = y_1\vec{w}_1 + \ldots + y_m\vec{w}_m \in W$, where $\vec{x} = (x_1, \ldots, x_n)^T$ are the coordinates of $\vec{v}$ **relative to the basis of** V and $\vec{y} = (y_1, \ldots, y_m)^T$ are those of $\vec{w}$ **relative to the basis of** W. Then, in those coordinates, there exists $\mathbf{S}^{m\times n}$ such that the linear function $\vec{w} = L[\vec{v}]$ is given by multiplication by $\mathbf{S}$, so $\vec{y} = \mathbf{S}\vec{x}$.

Proof: We mimic proof of Theorem 7.5 (previous section), in which we discovered the matrix representative by

$$L[\vec{x}] = \mathbf{A}\vec{x} = \left[L\left[\vec{e}_1\right] \ldots\ L\left[\vec{e}_n\right]\right]\vec{x} = x_1 L\left[\vec{e}_1\right] + \ \ldots\ + x_n L\left[\vec{e}_n\right].$$

This time, we'll be replacing the $\vec{e}_1$ with more general basis vectors.

In other words, we apply L to the basis vectors of V and express the result as a linear combination of the basis vectors in W. Byy linarity, we have:

$\vec{w} = L[\vec{v}] = L[x_1\vec{v}_1 + \ldots + x_n\vec{v}_n]$ (def. of $\vec{v}$)

$= x_1 L[\vec{v}_1] + \ldots + x_n L[\vec{v}_n]$ (linearity)

$= x_1 \sum_{i=1}^{m} s_{i1}\vec{w}_i + \ldots + x_n \sum_{i=1}^{m} s_{in}\vec{w}_i$ ($L[\vec{v}_i]$ live in W, and so linear combo of $\vec{w}_1, \ldots, \vec{w}_m$)

$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} s_{ij}\vec{w}_i = \sum_{j=1}^{n}\sum_{i=1}^{m} s_{ij}x_j\vec{w}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} s_{ij}x_j\right)\vec{w}_i,$ (arithmetic)

and so we find $y_i = \sum_{j=1}^{n} s_{ij}x_j$, and $\vec{y} = \mathbf{S}\vec{x}$, where $\mathbf{S} = [\vec{s}_1\ \ldots\ \vec{s}_n]$ and $\vec{s}_j = (s_{1j}, \ldots, s_{mj})$, as claimed. ■

Conversion: $\mathcal{B}$**-coords** $\longrightarrow$ **Std** (easy, always possible)

Example: Given $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$, basis of $\mathbb{R}^2$, find std coords for:

- $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ ...

$$\vec{x} = -3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 7\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -24 \end{bmatrix}.$$

- $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$ ...

$$\vec{x} = 8\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}.$$

In general, given $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$: $\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$

$$= \left[\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n\right]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{S}[\vec{x}]_{\mathcal{B}}.$$

So $\vec{x} = \mathbf{S}[\vec{x}]_{\mathcal{B}}$, **where S's columns are the basis vectors**.

Conversion: Std $\longrightarrow$ $\mathcal{B}$-coordinates (hard, not always possible)

Example: Again with $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$, find $\mathcal{B}$-coords for $\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

We know $\vec{x} = \mathbf{S}[\vec{x}]_{\mathcal{B}}$, i.e. $\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. ...

Method 1: $\left. \begin{aligned} c_1 + 2c_2 &= 7 \\ c_1 - 3c_2 &= 2 \end{aligned} \right\}$ Solve.

Method 2: $\vec{x} = \mathbf{S}[\vec{x}]_{\mathcal{B}} \quad \Rightarrow \quad [\vec{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\vec{x}$ ...

$$= -\tfrac{1}{5} \begin{bmatrix} -3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Example: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ form basis for $V = span(\vec{v}_1, \vec{v}_2) \subset \mathbb{R}^3$.

Find $[\vec{x}]_{\mathcal{B}}$ if $\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 10 \end{bmatrix}$ ...

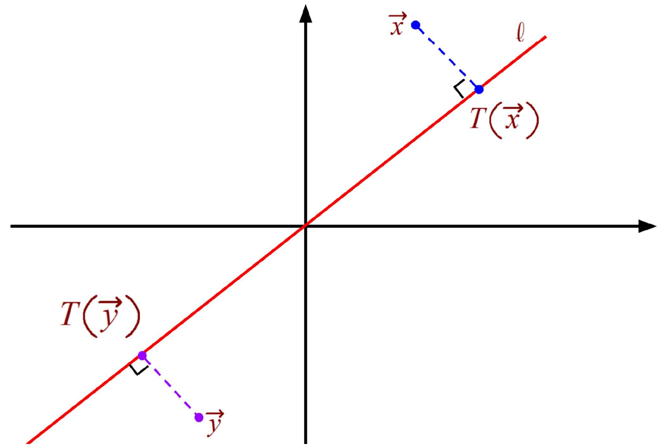
$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 10 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & 10 \end{array}\right] \quad \ldots$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 10 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 12 \end{array}\right] \qquad (!?!)$$

Resulting system has no solution. ($\vec{x} \notin V$) What does this mean?

How can $\mathcal{B}$-coordinates make my life easier?

Consider $T(\vec{x})$ = "projection of $\mathbb{R}^2$ onto line $\ell = \left\{(x,y) : y = \frac{3}{4}x\right\}$".



$T(\vec{x})$ = projection of 2D onto line $\ell = \left\{(x,y) : y = \frac{3}{4}x\right\}$

Turns out, $T(\vec{x}) = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}\vec{x}$. But why?

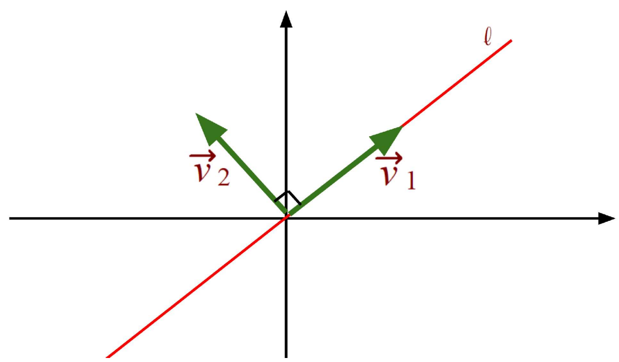
How would we find this? Let's find out.

Choose $\vec{v}_1$ to span ℓ, and $\vec{v}_2 \neq \vec{0}$ such that $\vec{v}_2 \perp \vec{v}_1$.

Example: $\vec{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ forms a basis of $\mathbb{R}^2$.

Any $\vec{x} \in \mathbb{R}^2$ can be written uniquely as combination of $\vec{v}_1, \vec{v}_2$.



$\vec{v}_1 \parallel \ell$ and $\vec{v}_2 \perp \vec{v}_1$

Notice that with this choice of vectors, $T(\vec{v}_1) = \vec{v}_1$, and $T(\vec{v}_2) = \vec{0}$.

So for any $\vec{x} \in \mathbb{R}^2$, $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = c_1\vec{v}_1 + 0\vec{v}_2 = c_1\vec{v}_1$.

Therefore, in $\mathcal{B}$-coords, we send $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ to $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$.

So in $\mathcal{B}$-coords, T is represented by $\mathbf{B}$ such that:

$$\mathbf{B}\vec{c} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}.$$

So, **B**, the $\mathcal{B}$-matrix of T, tells us how to perform T in $\mathcal{B}$-coords.

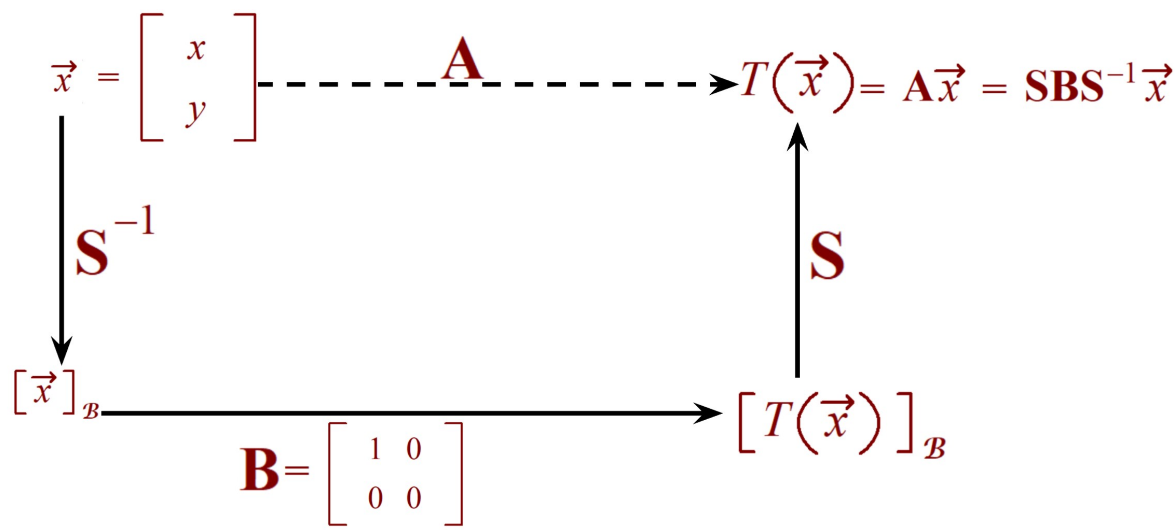
So we found the matrix! But not in the standard basis.

Q: How do we recover the "standard" $\mathbf{A} = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$ from the $\mathcal{B}$-matrix **S**?

Observe that: $\ell = span\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\mathcal{B} = \left\{\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right\}$, $\mathbf{S} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$, $\mathbf{S}^{-1} = \frac{1}{25}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$.

Using a "commutative diagram," we can construct the projection onto ℓ in pieces.

(commutative diagrams can be used to represent function composition)



We say a diagram **commutes** if all "paths" between some start/end points give the same overall function.

So $\mathbf{A} = \mathbf{SBS}^{-1}$.

For our line ℓ, $T(\vec{x}) = \mathbf{A}\vec{x} = \mathbf{SBS}^{-1}\vec{x} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{4}{25} & \frac{3}{25} \\ -\frac{3}{25} & \frac{4}{25} \end{bmatrix}\vec{x} = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}\vec{x}$. (we found it!)

❗ Reflection *across* ℓ would be $\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{4}{25} & \frac{3}{25} \\ -\frac{3}{25} & \frac{4}{25} \end{bmatrix}$ ($\vec{v}_2$ sent to $-\vec{v}_2$, not $\vec{0}$)

Definition (similar matrices): **A** is similar to **B**, written $\mathbf{A} \sim \mathbf{B}$, if they represent same linear transformation, but possibly in a different basis.

Algebraically: $\mathbf{A} = \mathbf{SBS}^{-1} \quad \Rightarrow \quad \mathbf{AS} = \mathbf{SB} \quad \Rightarrow \quad \mathbf{S}^{-1}\mathbf{AS} = \mathbf{B}$.

Example: From above: $\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Recall: Given linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$,

matrix representation of L is: $\mathbf{A} := \begin{bmatrix} | & | & & | \\ L(\vec{e}_1) & L(\vec{e}_2) & \ldots & L(\vec{e}_m) \\ | & | & & | \end{bmatrix}$.

Example: Find matrix form **B** of $L(x,y) = \begin{bmatrix} x-4y \\ -2x+3y \end{bmatrix}$ with respect to basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

$$\mathbf{S} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad \mathbf{A} = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}.$$

Therefore: $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}.$

Verifying: Let's characterize $\vec{e}_1, \vec{e}_2$ in $\mathcal{B}$-coords, apply **B**, and see if **B** send these vectors to the same location as **A** does.

Notice $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \mathbf{S}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = [\vec{e}_1]_{\mathcal{B}}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = [\vec{e}_2]_{\mathcal{B}}$.

Applying **B**, we get: $\mathbf{B}[\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix},$

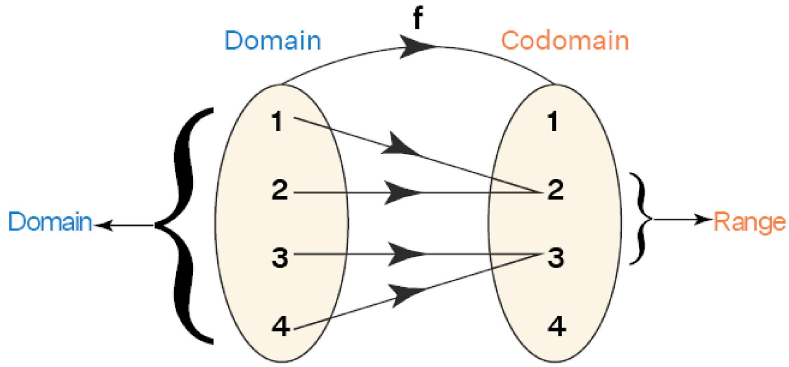
and $\mathbf{B}[\vec{e}_2]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix}.$

In std coords: $\mathbf{B}[\vec{e}_1]_{\mathcal{B}} = -\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{v}_1$

$\mathbf{B}[\vec{e}_2]_{\mathcal{B}} = -\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{7}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \vec{v}_2$ ✓

Videos explaining linear algebra with visually spectacular and intuitive explanations, see:
Youtube.com/watch?v=P2LTAUO1TdA&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Other Potentially Useful Materials (used in previous classes I've taught)



A function $f : X \to Y$ contains 3 pieces of information:

- Domain: X
- Codomain: Y
- A rule sending elements from domain to codomain: f

Example: $f(x) = x^2$. New notation: $f : \mathbb{R} \to \mathbb{R}$ where $x \mapsto x^2$ (x *maps* to x^2).

Example: $f : \mathbb{R}^3 \to \mathbb{R}^2$ where $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} \sin x \\ y+z \end{bmatrix}$.

Linear Transformation, a special type of function

Linear Transformation: $T : \mathbb{R}^m \to \mathbb{R}^n$ is a function which takes in values from $\mathbb{R}^m$ (vectors) and outputs values in $\mathbb{R}^n$ (vectors) and also satisfies $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$, for all $a, b \in \mathbb{R}$ and $\vec{u}, \vec{v} \in \mathbb{R}^m$.

Recall: A linear combination of vectors $\vec{u}, \vec{v}$ is a vector of the form $a\vec{u} + b\vec{v}$.

Linear transformations are functions which "preserve" linear combinations.

Example: Is $T : \mathbb{R}^2 \to \mathbb{R}^3$ where $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y-x \\ y \end{bmatrix}$ a linear transformation? ...

$$T\left(a\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)$$

$$= T\left(\begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} ax_1 + bx_2 \\ (ay_1 + by_2) - (ax_1 + bx_2) \\ ay_1 + by_2 \end{bmatrix}$$

$$= a\begin{bmatrix} x_1 \\ y_1 - x_1 \\ y_1 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 - x_2 \\ y_2 \end{bmatrix}$$

$$= aT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + bT\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right).$$

Example: Is $f(x) = 3x$ a linear transformation? ...

$$f(ax + by) = 3(ax + by) = 3ax + 3by = a3x + b3y = af(x) + bf(y).$$

Example: Is $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x^2 \\ 0 \end{bmatrix}$ a linear transformation? ...

Observe: $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

However: $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$!?!

Linear Transformations Theorem: $T : \mathbb{R}^m \to \mathbb{R}^n$ is **linear** if (and only if):

- $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$, for all $\vec{v}, \vec{w} \in \mathbb{R}^m$, and
- $T(k\vec{v}) = kT(\vec{v})$, for all $\vec{v} \in \mathbb{R}^m$ and all scalars k.

Example: Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{n \times m}$.

Observe: $\vec{x} = x_1\vec{e}_1 + \ldots + x_m\vec{e}_m$.

So, $\mathbf{A}\vec{x} = \mathbf{A}\left(x_1\vec{e}_1 + \ldots + x_m\vec{e}_m\right) = \mathbf{A}\left(x_1\vec{e}_1\right) + \ldots + \mathbf{A}\left(x_m\vec{e}_m\right) = x_1\mathbf{A}\left(\vec{e}_1\right) + \ldots + x_m\mathbf{A}\left(\vec{e}_m\right)$. (Chap. 1)

All information about linear transformations is encoded in where the transformation sends the basis vectors $\vec{e}_i$.

In $\mathbb{R}^2$, using standard basis:

$$\mathbf{A}\vec{e}_1 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \text{ and } \mathbf{A}\vec{e}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So, a linear transformation sends the 1st basis vector to the 1st column vector, and the 2nd basis vector to the 2nd column.

Matrix Columns of a Linear Transformation Theorem: Given linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$.

$$\text{Matrix of } T \text{ is: } \mathbf{A} = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \ldots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}.$$

Linear Transformation Matrix Correspondence: $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation $\Leftrightarrow$ $T(\vec{x}) = \mathbf{A}\vec{x}$, for some $\mathbf{A}^{n \times m}$.

Proof for $T : \mathbb{R}^2 \to \mathbb{R}^2$: First we show $\Rightarrow$.

$$\text{Suppose } T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

$$\text{Note if } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \mathbf{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \mathbf{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

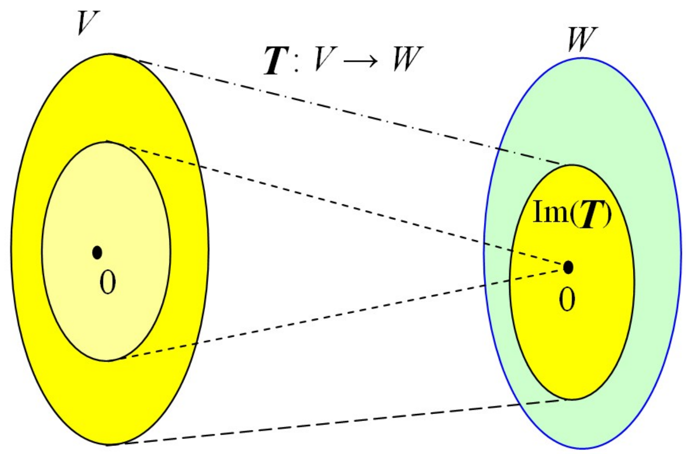
Now we show $\Leftarrow$.

Recall $\mathbf{A}\vec{x} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$ **is** a linear transformation (as seen earlier),

and its effect on $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the **same** as T.

So $\mathbf{A}\vec{x}$ is a linear transformation. ■

LT, Zero Identity Thm: If $T : \mathbb{R}^m \to \mathbb{R}^n$, then $T\left(\vec{0}\right) = \vec{0}$.



Proof: Let $\vec{0} \in \mathbb{R}^m$ and T be a linear transformation.

Then, $T\left(\vec{0}\right) =$ ...

$= T\left(0 \cdot \vec{0}\right)$

$= 0 \cdot T\left(\vec{0}\right) = \vec{0} \in \mathbb{R}^n$.

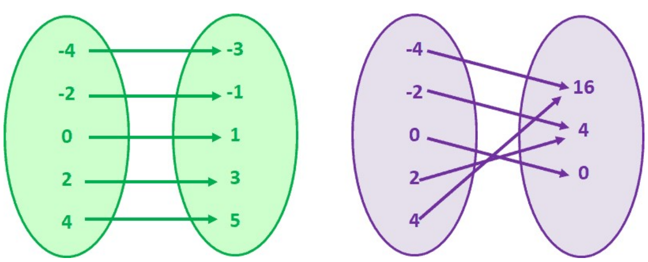
Alternatively: $T\left(\vec{0}\right)$

$= T(\vec{x} - \vec{x})$

$= T(\vec{x}) - T(\vec{x}) = \vec{0} \in \mathbb{R}^n$.

Corollary: The Contrapositive.

Invertibility



Invertible Transformation: $T : \mathbb{R}^m \to \mathbb{R}^n$ where there exists a map $S : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(S(\vec{x})) = \vec{x}$ and $S(T(\vec{y})) = \vec{y}$.

We usually denote the "inverse" S as T^{-1}.

Example: Show that $T : \mathbb{R} \to \mathbb{R}$ where $x \mapsto 3x$ and $S : \mathbb{R} \to \mathbb{R}$ where $x \mapsto \frac{1}{3}x$ are inverses. ...

$T(S(x))$

$= T\left(\frac{1}{3}x\right)$

$= 3 \cdot \left(\frac{1}{3}x\right) = x$ and ...

$S(T(x)) = S(3x)$

$= \frac{1}{3}(3x) = x.$ ■

Example: Does $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ x \end{bmatrix}$ have an inverse? ...

Observe that: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

If S exists, then $S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Whoops.

Practice

Which are linear transformations? Briefly justify.

a) $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+3y \\ x-y \\ y \end{bmatrix}$ b) $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+3y \\ x-y \\ y+1 \end{bmatrix}$ c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 3xy \\ 5yz \\ 7xz \end{bmatrix}$. ...

For each which is a linear transform, find the matrix representing it.

For a) : $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, so $\mathbf{A} = \begin{bmatrix} | & | \\ T\vec{e}_1 & T\vec{e}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$.