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Linearity

Up until now, we've exploited linearity mostly in $\mathbb{R}^n$, but now we'll generalize, showing its influence in other vector spaces. Indeed, this concept contributes to differential equations, integral equations, stochastic systems, and even quantum mechanics.

7.1 Linear Functions

Definition: Let V and W be real vector spaces. A function $L : V \to W$ is called *linear* if it obeys two basic rules:

$L[\vec{v} + \vec{w}] = L[\vec{v}] + L[\vec{w}]$, $L[c\vec{v}] = cL[\vec{v}]$, for all $\vec{v}, \vec{w} \in V$ and all scalar c.

We will call V the domain and W the codomain for L.

Or more succinctly, a linear function respects linear combinations:

$L[c\vec{v} + d\vec{w}] = cL[\vec{v}] + dL[\vec{w}]$, for all $\vec{v}, \vec{w} \in V$, $c, d \in \mathbb{R}$. $(*)$

Algebraic Linear Functions

Zero function: $O[\vec{v}] = \vec{0}$.

Identity Function: $I[\vec{v}] = \vec{v}$.

Scalar Multiplication: $M_a[\vec{v}] = a\vec{v}$ for $a \in \mathbb{R}$.

NOT Linear Functions: Even though $y = L[x] = ax + b$ is a straight line, it is not a linear fn when $b \neq 0$.

Indeed, $L[x_1 + x_2] = a(x_1 + x_2) + b$

$\neq (ax_1 + b) + (ax_2 + b) = L[x_1] + L[x_2]$.

Also: $L[0] = b \neq 0$.

Instead, these are called *affine functions*.

Matrices: Matrices are easily seen as linear functions.

Observe that $\mathbf{A}[c\vec{v} + d\vec{w}] = c\mathbf{A}\vec{v} + d\mathbf{A}\vec{w}$.

In fact:

Thm: *Every* linear function $L : \mathbb{R}^n \to \mathbb{R}^m$ (between finite dim spaces) is given by matrix multiplication: $L[\vec{v}] = \mathbf{A}^{m\times n}\vec{v}$. (!?!)

Proof: Key idea: what does linear function do to basis vectors? Let $\widehat{e}_1, \dots, \widehat{e}_n$ be standard basis of $\mathbb{R}^n$, and let $\widehat{e}'_1, \dots, \widehat{e}'_m$ be standard basis of $\mathbb{R}^m$. Since $L[\widehat{e}_j] \in \mathbb{R}^m$, we can write it as a linear combination of the latter basis vectors:

$$L[\widehat{e}_j] = \vec{a}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = a_{1j}\widehat{e}'_1 + \ldots + a_{mj}\widehat{e}'_m, \quad j = 1, \dots, n. \qquad (**)$$

Let's construct: $\mathbf{A}^{m \times n} = \left(\vec{a}_1 \ \dots \ \vec{a}_n\right) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, Whose columns are the image vectors $(**)$.

Using $(*)$, we then compute the effect of L on a general vector $\vec{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$:

$$L[\vec{v}] = L[v_1\widehat{e}_1 + \ldots + v_n\widehat{e}_n] = v_1 L[\widehat{e}_1] + \ldots + v_n L[\widehat{e}_n] = v_1\vec{a}_1 + \ldots + v_n\vec{a}_n = \mathbf{A}\vec{v}.$$

The final inequality follows from our basic formula: $\mathbf{A}\vec{c} = c_1\vec{a}_1 + \ldots + c_k\vec{a}_k$.

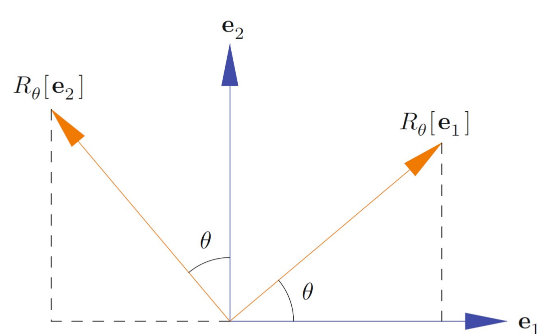
We conclude that the vector $L[\vec{v}]$ coincides with the vector $\mathbf{A}\vec{v}$. ■

This proof shows how to construct the matrix representative of a linear function L. That is: $\mathbf{A} = \left[L[\widehat{e}_1] \ \dots \ L[\widehat{e}_n]\right]$.

Example: Rotations! Let's find matrix representative of rotating 2D space by θ.

According to the proof above, we want:

$$L[\vec{v}] = \mathbf{A}\vec{v} = \left[L[\widehat{e}_1] \quad L[\widehat{e}_2]\right]\vec{v}.$$



And note that rotating $\widehat{e}_1$ by θ gives $(\cos\theta)\widehat{e}_1 + (\sin\theta)\widehat{e}_2 = (\cos\theta, \sin\theta)$.

rotating $\widehat{e}_2$ by θ gives $-(\sin\theta)\widehat{e}_1 + (\cos\theta)\widehat{e}_2 = (-\sin\theta, \cos\theta)$.

Therefore: $\mathbf{A}\vec{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{v}$.

Example: Explain why $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\vec{F}(\vec{x}) = \begin{bmatrix} x + e^y \\ 2x + y \end{bmatrix}$ is not linear.

$$\vec{F}\left(\vec{0}\right) = \begin{bmatrix} 0 + e^0 \\ 2(0) + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \vec{F}(c\vec{x}) = \begin{bmatrix} cx + e^{cy} \\ 2cx + cy \end{bmatrix} \neq \begin{bmatrix} cx + ce^y \\ 2cx + cy \end{bmatrix} = c\vec{F}(\vec{x}).$$

Example: Explain why the translation function $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$ for $a, b \in \mathbb{R}$, is almost never linear. Precisely when is it linear?

$$T\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix}\right) = \begin{bmatrix} c_1x_1 + c_2x_2 + a \\ c_1y_1 + c_2y_2 + b \end{bmatrix} \neq \begin{bmatrix} c_1x_1 + c_2x_2 + 2a \\ c_1y_1 + c_2y_2 + 2b \end{bmatrix}$$

$$= \begin{bmatrix} c_1x_1 + a \\ c_1y_1 + b \end{bmatrix} + \begin{bmatrix} c_2x_2 + a \\ c_2y_2 + b \end{bmatrix} = c_1T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \qquad \text{(unless } a = b = 0\text{)}.$$

Linear Operators

Functions spaces: Much wider variety of linear operators available (not just matrices). Complete classification is out of the question. But let's look at some common examples:

Example: Evaluation of a Function: For $f \in C^0[a,b]$ operate as $L[f] = f(x_0)$ with $L : C^0[a,b] \to \mathbb{R}$.

$$L[cf + dg] = (cf + dg)(x_0) = cf(x_0) + dg(x_0) = cL[f] + dL[g].$$

Example: Integration operator on $f \in C^0[a,b]$ as $\mathfrak{I}[f] = \int_a^b f(x)dx$ with $\mathfrak{I} : C^0[a,b] \to \mathbb{R}$.

$$\mathfrak{I}[cf + dg] = \int_a^b (cf(x) + dg(x))dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx = c\mathfrak{I}[f] + d\mathfrak{I}[g].$$

Example: Indefinite Integral operator on $f \in C^0[a,b]$ as $J[f] = \int_a^x f(x)dx$ with $J : C^0[a,b] \to C^0[a,b]$.

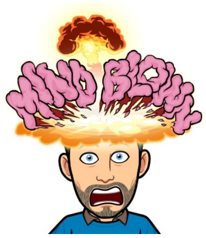
$$J[cf + dg] = \int_a^x (cf(x) + dg(x))dx = c\int_a^x f(x)dx + d\int_a^x g(x)dx = cJ[f] + dJ[g].$$

Example: Differentiation operator on $f \in C^1[a,b]$ as $D[f] = f'$ with $D : C^1[a,b] \to C^0[a,b]$.

$$D[cf + dg] = (cf(x) + dg(x))' = cf'(x) + dg'(x) = cD[f] + dD[g].$$

The Space of Linear Functions

Let's take linearity up a notch! Given two vector spaces V, W, let's use $\mathcal{L}(V, W)$ to denote set of all linear functions $L : V \to W$.



It turns out $\mathcal{L}(V, W)$ (in addition to operating on vector spaces) is *itself,* a vector space!

Addition Notation/Definition: $(L+M)[\vec{v}] := L[\vec{v}] + M[\vec{v}]$.

So is $L+M$ linear? $\quad (L+M)[a\vec{v}+b\vec{w}] = L[a\vec{v}+b\vec{w}] + M[a\vec{v}+b\vec{w}]$ (definition above)

$= aL[\vec{v}] + bL[\vec{w}] + aM[\vec{v}] + bM[\vec{w}]$ (linearity of L, M)

$= (aL[\vec{v}] + aM[\vec{v}]) + (bL[\vec{w}] + bM[\vec{w}])$

$= a(L+M)[\vec{v}] + b(L+M)[\vec{w}]$. (definition above) So, yes. $L+M$ is linear.

Scalar Mult. Notation/Definition: $(cL)[\vec{v}] := cL[\vec{v}]$. Linearity shown similarly to above.

Zero Function: $O[\vec{v}] :\equiv \vec{0}$. Vector field axioms are satisfied with these definitions (check them!).

Concretely: Space of linear transformations in the plane: $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is identified with space $\mathcal{M}_{2\times 2}$ of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Standard basis: $\mathbf{E}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{E}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{E}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{E}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Every matrix *uniquely* written as: $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\mathbf{E}_{11} + b\mathbf{E}_{12} + c\mathbf{E}_{21} + d\mathbf{E}_{22}$.

Dual Spaces

Definition: The *dual space to a vector space V* is the vector space $V^* = \mathcal{L}(V, \mathbb{R})$ consisting of all real valued linear functions $\ell : V \to \mathbb{R}$.

If $V = \mathbb{R}^n$, then, by previous them, every linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ is given by multiplication by $1 \times n$ matrix, i.e., a row vector.
$\ell[\vec{v}] = \vec{a}^T\vec{v} = a_1v_1 + \ldots + a_nv_n$.

Therefore, we can identify the dual space $(\mathbb{R}^n)^*$ with the space of row vectors with n entries.

Row vectors should more properly be viewed as real valued linear functions, the dual objects to column vectors.

Theorem: Let V be a finite dimensional real inner product space. Then every linear function $\ell : V \to \mathbb{R}$ is given by taking the inner product with a fixed vector $\vec{a} \in V$: $\ell[\vec{v}] = \langle \vec{a}, \vec{v} \rangle$.

Proof: Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis of V. If we write $\vec{v} = y_1\vec{v}_1 + \ldots + y_n\vec{v}_n$, then, by linearity,

$$\ell[\vec{v}] = y_1\ell[\vec{v}_1] + \ldots + y_n\ell[\vec{v}_n] = b_1y_1 + \ldots + b_ny_n, \text{ where } b_i = \ell[\vec{u}_i]. \qquad (*)$$

On the other hand, if we write $\vec{a} = x_1\vec{v}_1 + \ldots + x_n\vec{v}_n$, then

$$\langle \vec{a}, \vec{v} \rangle = \sum_{i,j=1}^{n} x_jy_i\langle \vec{v}_i, \vec{v}_j \rangle = \sum_{i,j=1}^{n} g_{ij}x_jy_i, \qquad (**)$$

where $G = (g_{ij})$ is the $n \times n$ Gram matrix with entries $g_{ij} = \langle \vec{v}_i, \vec{v}_j \rangle$.

Equality of $(*)$ and $(**)$ requires that $G\vec{x} = \vec{b}$, where $\vec{x} = (x_1, \ldots, x_n)^T$, $\vec{b} = (b_1, \ldots, b_n)^T$.

Invertibility of G is guaranteed by Theorem 3.34 allows us to solve for $\vec{x} = G^{-1}\vec{b}$ and thereby construct the desired $\vec{a}$.

In particular, if $\vec{v}_1, \ldots, \vec{v}_n$ is an orthonormal basis, then $G = 1$ and hence $\vec{a} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$. ■

Example: Write down a basis for, and dimension of, the linear function space $\mathcal{L}(P^{(3)}, \mathbb{R})$.

This is 4D. In particular, if you view the polynomials as vectors in $\mathbb{R}^4$, the polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$ can be written (a_3, a_2, a_1, a_0), with basis $L_0 = (0,0,0,1)^T$, $L_1 = (0,0,1,0)^T$, etc.

Example: Given a basis $\vec{v}_1, \ldots, \vec{v}_n$ of V, let the dual basis $\ell_1, \ldots, \ell_n$ of V^* consists of the linear functions uniquely defined by the requirements $\ell_i(\vec{v}_j) = \begin{cases} 1 \text{ if } & i = j, \\ 0 \text{ if } & i \neq j. \end{cases}$ Find the dual basis for: $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (1,0,1)$, and $\vec{v}_3 = (0,1,1)$.

Note we were given column vectors. However, we are looking for elements ℓ_i of $(\mathbb{R}^3)^*$ (row vectors) such that

$$\ell_i(\vec{v}_j) = \begin{cases} 1 \text{ if } & i = j, \\ 0 \text{ if } & i \neq j. \end{cases}$$

Let's require $\ell_1(\vec{v}_1) = \ell_1((1,1,0)) = (\ell_{11}, \ell_{12}, \ell_{13})^T(1,1,0) = 1$.
And similarly that $(\ell_{11}, \ell_{12}, \ell_{13})^T(1,0,1) = (\ell_{11}, \ell_{12}, \ell_{13})^T(0,1,1) = 0$.

From this we get: $\ell_{11} + \ell_{12} = 1, \qquad \ell_{11} + \ell_{13} = 0, \qquad \ell_{12} + \ell_{13} = 0.$

$\Rightarrow$ From the 2nd two eqs: $\ell_{11} = -\ell_{13}, \quad \ell_{12} = -\ell_{13}$, so $\ell_{11} = \ell_{12} = -\ell_{13}$.

$\Rightarrow$ From the 1st eq: $\ell_{11} + \ell_{12} = 2\ell_{11} = 1$ or $\ell_{11} = \frac{1}{2}$. Therefore: $(\ell_{11}, \ell_{12}, \ell_{13}) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$.

Let's require $\ell_2(\vec{v}_2) = \ell_2((1,0,1)) = (\ell_{21}, \ell_{22}, \ell_{23})^T(1,0,1) = 1$.
And similarly that $(\ell_{21}, \ell_{22}, \ell_{23})^T(1,1,0) = (\ell_{21}, \ell_{22}, \ell_{23})^T(0,1,1) = 0$.

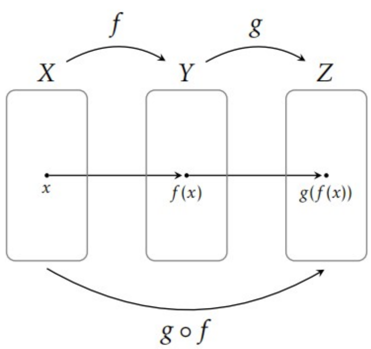
From this we get: $\ell_{21} + \ell_{23} = 1, \quad \ell_{21} + \ell_{22} = 0, \quad \ell_{22} + \ell_{23} = 0$.

$\Rightarrow$ $\ell_{21} = \ell_{23} = -\ell_{22}$ and $\ell_{21} = \frac{1}{2}$. Therefore: $(\ell_{21}, \ell_{22}, \ell_{23}) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

Similarly, we find $(\ell_{31}, \ell_{32}, \ell_{33}) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Therefore, the dual basis is $\left\{\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}$.

Composition



So far, we've added linear functions, and multiplied them by scalars.
What about composing them?

Lemma: Let V, W, Z be vector spaces. If $L : V \to W$ and $M : W \to Z$ are linear functions, then the *composite function* $M \circ L : V \to Z$, defined by $(M \circ L)[\vec{v}] := M[L[\vec{v}]]$ is also linear.

Proof: $(M \circ L)[c\vec{v} + d\vec{w}] = M[L[c\vec{v} + d\vec{w}]]$ (definition)

$= M[cL[\vec{v}] + dL[\vec{w}]]$ (linearity of L)

$= cM[L[\vec{v}]] + dM[L[\vec{w}]]$ (linearity of M)

$= c(M \circ L)[\vec{v}] + d(M \circ L)[\vec{w}]$ (definition) ■

Concretely: The *commutator* of two linear transformations $L, M : V \to V$ on a vector space V is:
$K := [L, M] := L \circ M - M \circ L$, where $[L, M]$ is referred to as a *Lie bracket*.

Prove that the commutator K is a linear transformation on V.

$K[a\vec{v} + b\vec{w}] = [L, M][a\vec{v} + b\vec{w}] = (L \circ M - M \circ L)[a\vec{v} + b\vec{w}]$ (definition of K)

$= L[M[a\vec{v} + b\vec{w}]] - M[L[a\vec{v} + b\vec{w}]]$ (definition of composition)

$= L[aM[\vec{v}] + bM[\vec{w}]] - M[aL[\vec{v}] - bL[\vec{w}]]$ (linearity of L, M)

$= aL[M[\vec{v}]] + bL[M[\vec{w}]] - aM[L[\vec{v}]] - bM[L[\vec{w}]]$ (linearity of L, M)

$= (aL[M[\vec{v}]] - aM[L[\vec{v}]]) + (bL[M[\vec{w}]] - bM[L[\vec{w}]])$ (algebra)

$= a(L \circ M[\vec{v}] - M \circ L[\vec{v}]) + b(L \circ M[\vec{w}] - M \circ L[\vec{w}])$ (definition of composition)

$= a(L \circ M - M \circ L)[\vec{v}] + b(L \circ M - M \circ L)[\vec{w}] = a[L,M][\vec{v}] + b[L,M][\vec{w}]$ (definition of K)

$= aK[\vec{v}] + bK[\vec{w}].$ (definition of K)

The Jacoby Identity: $[[L,M],N] + [[N,L],M] + [[M,N],L] = O.$

Applications: **Cross product** $\vec{a} \times \vec{b}$ and **Lie bracket** operation $[L,M]$ satisfy the Jacobi identity. In **analytical mechanics**, the Jacobi identity is satisfied by the **Poisson brackets** $\{f,g\}$. In **quantum mechanics**, it is satisfied by operator commutators on a Hilbert space and equivalently in the **phase space** formulation of quantum mechanics by the **Moyal bracket** $\{\{f,g\}\}$.

Prove that the Jacoby Identity: $[[L,M],N] + [[N,L],M] + [[M,N],L] = O$ is valid for any three transformations.

$[[L,M],N] = [L,M] \circ N - N \circ [L,M] = (L \circ M - M \circ L) \circ N - N \circ (L \circ M - M \circ L)$
$= (L \circ M \circ N - M \circ L \circ N) - (N \circ L \circ M - N \circ M \circ L) = L \circ M \circ N - M \circ L \circ N - N \circ L \circ M + N \circ M \circ L$

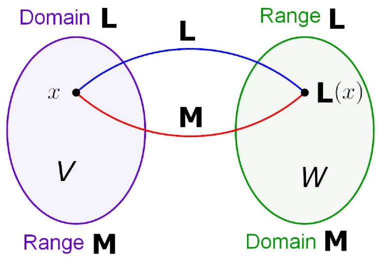
Similarly: $[[N,L],M] = N \circ L \circ M - L \circ N \circ M - M \circ N \circ L + M \circ L \circ N,$
and $[[M,N],L] = M \circ N \circ L - N \circ M \circ L - L \circ M \circ N + L \circ N \circ M.$

Therefore: $[[L,M],N] + [[N,L],M] + [[M,N],L]$
$= (L \circ M \circ N - M \circ L \circ N - N \circ L \circ M + N \circ M \circ L) + (N \circ L \circ M - L \circ N \circ M - M \circ N \circ L + M \circ L \circ N)$
$+ (M \circ N \circ L - N \circ M \circ L - L \circ M \circ N + L \circ N \circ M)$

$= (L \circ M \circ N - L \circ M \circ N + L \circ N \circ M - L \circ N \circ M) + (N \circ L \circ M - N \circ L \circ M + N \circ M \circ L - N \circ M \circ L)$
$+ (M \circ N \circ L - M \circ N \circ L + M \circ L \circ N - M \circ L \circ N)$

$= O.$ ■

Inverses



Do linear functions have inverses? We've seen that square matrices do.

More generally, we have the following:

Definition: Let $L : V \to W$ be a linear function. If $M : W \to V$ is a function such that:

$$L \circ M = I_W, \qquad M \circ L = I_V, \qquad (***)$$

are equal to the identity function (in their respective spaces), then we call M the *inverse* of L and write $M = L^{-1}$.

In other words, being an inverse requires: $L[M[\vec{w}]] = \vec{w}$ for all $\vec{w} \in W$, and $M[L[\vec{v}]] = \vec{v}$ for all $\vec{v} \in V$.

When it exists, the inverse is unique (proof part of book exercise)!

Informally: Observe that since we relabeled the right- and left-inverse (M) of L as L^{-1} in $(***)$, then $(L^{-1})^{-1} = L$.

Lemma: If it exists, the inverse of a linear function is also a linear function.

Proof: Let L, M satisfy the conditions of the previous definition.

Given $\vec{w}, \vec{w}' \in W$. We wish to show that M is linear, or (given scalars c, d) that $M\left[c\vec{w} + d\vec{w}'\right] = cM[\vec{w}] + dM\left[\vec{w}'\right]$.

Since M and L are inverses, we note $\vec{w} = (L \circ M)[\vec{w}] = L[M[\vec{w}]]$,

and $\vec{w}' = (L \circ M)\left[\vec{w}'\right] = L\left[M\left[\vec{w}'\right]\right]$.

Therefore, using only linearity of L :

$$M\left[c\vec{w} + d\vec{w}'\right] = M\left[cL[M[\vec{w}]] + dL\left[M\left[\vec{w}'\right]\right]\right] \qquad \text{(from above)}$$

$$= M\left[L\left[cM[\vec{w}] + dM\left[\vec{w}'\right]\right]\right] \qquad \text{(linearity of } L\text{)}$$

$$= (M \circ L)\left[cM[\vec{w}] + dM\left[\vec{w}'\right]\right] \qquad \text{(definition of composition)}$$

$$= cM[\vec{w}] + dM\left[\vec{w}'\right], \qquad (M \text{ and } L \text{ are inverses})$$

proving linearity of M. ■

On finite dimensional domains, since linear transformations can be written as matrices, this results in a very simple situation.

That is, recall that if $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, such that L, M are given by matrix multiplication by $\mathbf{A}$ and $\mathbf{B}$ respectively, then the inverse definition's conditions $(***)$ reduce to the usual conditions: $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$. In particular, $\mathbf{I}$ is an $n \times n$ matrix, requiring $m = n$, and for its coefficient matrix $\mathbf{A}$ to be nonsingular.

However, on infinite dimensional function spaces, things are more subtle.

Concretely: Let $D[f] = f'$ represent differentiation on $C^1[a,b]$, and
$J[f] = g$ represent integration, where $g(x) = \int_a^x f(y)dy$ on $C^0[a,b]$.

We often consider these "inverse" operations.

Indeed, observe that $(D \circ J)[f] = D[J[f]] = D[g] = g' = f$, since

$$g'(x) = \frac{d}{dx}\int_a^x f(y)\,dy = f(x). \qquad \text{(Fund. Thm. Calc)}$$

However, what about $(J \circ D)[f]$? This is $J[D[f]] = J\left[f'\right] = h$, where

$$h(x) = \int_a^x f'(y)dy = f(x) - f(a).$$

So $h(x) \neq f(x)$, unless $f(a) = 0$. So, $J \circ D \neq I_{C^1[a,b]}$.

In other words, D is a left-inverse for J, but not a right-inverse!

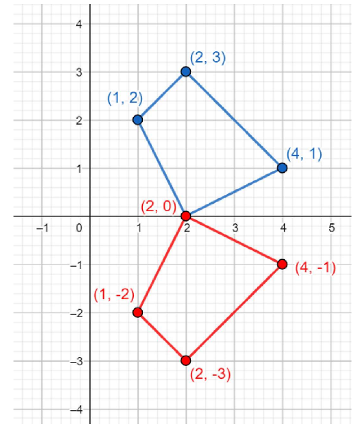
If we restrict D to $V := \{f : f(a) = 0\} \subset C^1[a,b]$, then an inverse is defined (as seen above).

But notice that $V \subsetneq C^1[a,b] \subsetneq C^0[a,b]$.

So J defines a one-to-one invertible, linear map from a vector space $C^0[a,b]$ to a proper subspace $V \subsetneq C^0[a,b]$!

This can't happen in finite dimensions. A matrix (linear map) is invertible only when the size of it's image is the same as it's domain and codomain.

Concretely: Determine if linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by reflection through the x-axis has an inverse. If so, describe it.



How does this transformation $L\langle x,y\rangle \to \langle x,-y\rangle$ affect the basis vectors? ...

$$L\langle 1,0\rangle \to \langle 1,0\rangle, \qquad L\langle 0,1\rangle \to \langle 0,-1\rangle.$$

Therefore: $L\langle x,y\rangle = \mathbf{A}\vec{x} = \begin{bmatrix} L(\hat{e}_1) & L(\hat{e}_2) \end{bmatrix}\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\vec{x}.$

And observe that $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{A}.$

So, yes L has an inverse. And it is its *own* inverse (Which intuitively/geometrically makes sense)!