Applied Linear Algebra

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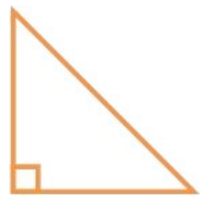
Chapter 4: Orthogonality



$\hat{e}_1 \perp \hat{e}_2$

Orthogonality is the generalization of the idea of perpendicularity.   Recall $\vec{v}, \vec{w}$ are called orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$.

Definition: A basis $\vec{u}_1, \ldots, \vec{u}_n$ of an n-dimensional inner product space V is called an *orthogonal basis* if $\langle \vec{u}_i, \vec{u}_j \rangle = 0$ for all $i \neq j$. It is called an *orthonormal basis* if, in addition, each vector has unit length: $|\vec{u}_i| = 1$, for all $i = 1, \ldots, n$.

In application, and in theoretical research, choosing mutually orthogonal basis elements is extremely powerful.

Definition: **A** is called an *orthogonal matrix* if its columns form an orthonormal system.

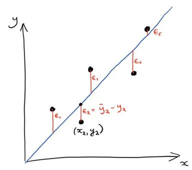
In this chapter, we will learn how to adjust a basis to be an **orthonormal basis** (§4.2, Gram-Schmidt), to decompose matrices into **orthogonal matrices** (§4.3 QR decomp), and to **orthogonally project vectors** onto subspaces (§4.4). We will also learn about **orthogonal subspaces** which are orthogonal to each other (§4.4).

The four fundamental subspaces of a matrix ((co-)image,(co-)kernel) that were introduced in §2 come in mutually orthogonal pairs. This will lead to an intriguing result in §4.4 (Fredholm Alternative).

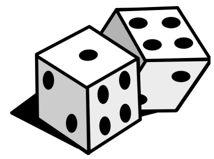
Applications

Fourier analysis decomposes waves (music, or *any* signal) into sines/cosines which form orthogonal system of functions (CDs, DVDs, MP3s).

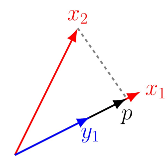
Orthogonal projections turn out to be incredibly important in:



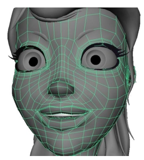
Statistics (regression)



Probability

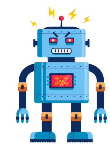


Gram-Schmidt



Computer graphics

Data science/analysis



Pattern recognition/AI

4.1 Orthogonal and Orthonormal Bases

Lemma: If $\vec{v}_1, \ldots, \vec{v}_n$ is an orthogonal basis of a vector space V, then the normalized vectors
$\vec{u}_i = \frac{\vec{v}_i}{|\vec{v}_i|}$, $i = 1, \ldots, n$ form an orthonormal basis.

Specifically: $\vec{v}_1 = (1,2,-1)$, $\vec{v}_2 = (0,1,2)$, and $\vec{v}_3 = (5,-2,1)$ are easily seen to form a basis of $\mathbb{R}^3$.

One can also check: $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$. So, form an orthogonal basis wrt standard dot product.

When we divide each orthogonal basis by its length, the result is the orthonormal basis:
$\vec{u}_1 = \frac{1}{\sqrt{6}}(1,2,-1) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$, $\vec{u}_2 = \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, and $\vec{u}_3 = \left(\frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$.

Observe $|\vec{u}_i| = 1$, and $\vec{u}_i \cdot \vec{u}_j = 0$, when $i \neq j$.

Proposition: Let $\vec{v}_1, \ldots, \vec{v}_n \in V$ be nonzero, mutually orthogonal elements, so $\vec{v}_i \neq \vec{0}$ and $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for all $i \neq j$.
Then $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

Proof: Suppose $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$.

We must show that the c_i need be zero.

Let's take the inner product of this equation with any $\vec{v}_i$. Using linearity and orthogonality, we compute:
$0 = \langle c_1\vec{v}_1 + \ldots + c_k\vec{v}_k, \vec{v}_i \rangle = c_1\langle \vec{v}_1, \vec{v}_i \rangle + \ldots + c_k\langle \vec{v}_k, \vec{v}_i \rangle = c_i\langle \vec{v}_i, \vec{v}_i \rangle = c_i|\vec{v}_i|^2$.

Therefore, given that $\vec{v}_i \neq \vec{0}$, we conclude that $c_i = 0$.

Since this holds for all $i = 1,\dots,k$, the linear independence of $\vec{v}_1,\dots,\vec{v}_n$ follows. ■

Theorem: Suppose $\vec{v}_1,\dots,\vec{v}_n$ are nonzero, mutually orthogonal elements of an inner product space V.
Then, $\vec{v}_1,\dots,\vec{v}_n$ form an orthogonal basis for their span $W = span\{\vec{v}_1,\dots,\vec{v}_n\} \subset V$, which is therefore a subspace of dimension $n = \dim W$. In particular, if $\dim V = n$, then $\vec{v}_1,\dots,\vec{v}_n$ form an orthogonal basis for V.

Specifically: Consider $P^{(2)}$, consisting of $p(x) = \alpha + \beta x + \gamma x^2$, equipped with the L^2 inner product and norm.

Standard monomials $1,x,x^2$ do *not* form an orthogonal basis. As calculated previously:
$\langle 1,x\rangle = \frac{1}{2}, \quad \langle 1,x^2\rangle = \frac{1}{3}, \quad \langle x,x^2\rangle = \frac{1}{4}.$

One orthogonal basis is provided by: $p_1(x) = 1, \quad p_2(x) = x - \frac{1}{2}, \quad p_3(x) = x^2 - x + \frac{1}{6}.$

One easily verifies $\langle p_1,p_2\rangle = \langle p_1,p_3\rangle = \langle p_2,p_3\rangle = 0$, while $\|p_1\| = 1, \quad \|p_2\| = \frac{1}{2\sqrt{3}}, \quad \|p_3\| = \frac{1}{6\sqrt{5}}.$

Forming the orthonormal basis: $u_1(x) = 1, \quad u_2(x) = \sqrt{3}\,(2x-1), \quad u_3(x) = \sqrt{5}\,(6x^2 - 6x + 1).$

§4.5 will give us a systematic way to find the orthogonal basis yourself.

Computations in Orthogonal Bases

In high dimensions, computations can take a long time. However, switching to an orthogonal or orthonormal system can dramatically speed up the computations. This has allowed for many advancements including MP3s, CDs, DVDs, YouTube, least-squares approximations, and the statistical analysis of large data sets (and much more).

Theorem: Let $\vec{u}_1,\dots,\vec{u}_n$ be an orthonormal basis for an inner product space V. Then, one can write any element $\vec{v} \in V$ as a linear combination: $\vec{v} = c_1\vec{u}_1 + \dots + c_n\vec{u}_n$, in which its coordinates can be calculated as $c_i = \langle \vec{v},\vec{u}_i\rangle$.
Moreover, $\vec{v}$'s norm is given by the Pythagorean formula: $\|\vec{v}\| = \sqrt{c_1^2 + \dots + c_n^2} = \sqrt{\sum_{i=1}^{n}\langle \vec{v},\vec{u}_i\rangle^2}$.

Proof: Let's compute the inner product of $\vec{v}$ (or $c_1\vec{u}_1 + \dots + c_n\vec{u}_n$) with one of the basis vectors.

Using the orthonormality conditions: $\langle \vec{u}_i,\vec{u}_j\rangle = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$ and bilinearity of the inner product, we obtain:

$$\langle \vec{v},\vec{u}_i\rangle = \left\langle \sum_{j=1}^{n} c_j\vec{u}_j,\vec{u}_i\right\rangle = \sum_{j=1}^{n} c_j\langle \vec{u}_j,\vec{u}_i\rangle = c_i\|\vec{u}_i\|^2 = c_i.$$

To prove the last formula in the theorem, we similarly expand

$$|\vec{v}|^2 = \langle \vec{v}, \vec{v} \rangle = \left\langle \sum_{j=1}^n c_i \vec{u}_i, \sum_{j=1}^n c_j \vec{u}_j \right\rangle = \sum_{i,j=1}^n c_i c_j \langle \vec{u}_i, \vec{u}_j \rangle = \sum_{i=1}^n c_i^2,$$

again making use of the orthonormality of the basis elements. ■

Specifically: let's rewrite $\vec{v} = (1,1,1)$ in terms of the orthonormal basis:

$$\vec{u}_1 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), \quad \vec{u}_2 = \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \text{ and } \vec{u}_3 = \left(\frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right).$$

Computing the dot product: $\vec{v} \cdot \vec{u}_1 = \frac{2}{\sqrt{6}}$, $\vec{v} \cdot \vec{u}_2 = \frac{3}{\sqrt{5}}$, $\vec{v} \cdot \vec{u}_3 = \frac{4}{\sqrt{30}}$. And therefore:

$$\vec{v} = \frac{2}{\sqrt{6}}\vec{u}_1 + \frac{3}{\sqrt{5}}\vec{u}_2 + \frac{4}{\sqrt{30}}\vec{u}_3.$$

Observe how much easier this is than solving the linear system $\vec{v} = [\vec{u}_1 \ldots \vec{u}_n]\vec{c}$ (chapter 2).

Theorem: If $\vec{v}_1, \ldots, \vec{v}_n$ form an orthogonal basis, then the corresponding coordinates of $\vec{v} := a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$ are given by $a_i = \frac{\langle \vec{v}, \vec{v}_i \rangle}{|\vec{v}_i|^2}$. In this case, its norm can be computed using the formula: $|\vec{v}|^2 = \sum_{i=1}^n a_i^2 |\vec{v}_i|^2 = \sum_{i=1}^n \left(\frac{\langle \vec{v}, \vec{v}_i \rangle}{|\vec{v}_i|}\right)^2$.

Example: Let $\mathbb{R}^2$ have the inner product defined by the positive definite matrix $\mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$.

a) Show that $\vec{v}_1 = (1,1)$, $\vec{v}_2 = (-2,1)$ form an orthogonal basis. ...

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^T \mathbf{K} \vec{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = 0.$$

b) Write $\vec{v} = (3,2)$ as a linear combination of $\vec{v}_1, \vec{v}_2$ using the orthogonality formul in the above theorem.

$$a_1 = \frac{\langle \vec{v}, \vec{v}_1 \rangle}{|\vec{v}_1|^2}, \qquad a_2 = \frac{\langle \vec{v}, \vec{v}_2 \rangle}{|\vec{v}_2|^2}.$$

$$\langle \vec{v}, \vec{v}_1 \rangle = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7. \qquad \langle \vec{v}, \vec{v}_2 \rangle = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -5.$$

$$|\vec{v}_1|^2 = \langle \vec{v}_1, \vec{v}_1 \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3. \qquad |\vec{v}_2|^2 = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 15.$$

So, $a_1 = \frac{7}{3}$, $a_2 = \frac{-5}{15}$ and $\vec{v} = \frac{7}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2$.

c) Verify the norm formula in the above theorem for $|\vec{v}|$.

$$|\vec{v}|^2 = \sum_{i=1}^{n} a_i^2 |\vec{v}_i|^2 = \sum_{i=1}^{n} \left(\frac{\langle \vec{v}, \vec{v}_i \rangle}{|\vec{v}_i|} \right)^2 = \left(\frac{\langle \vec{v}, \vec{v}_1 \rangle}{|\vec{v}_1|} \right)^2 + \left(\frac{\langle \vec{v}, \vec{v}_2 \rangle}{|\vec{v}_2|} \right)^2 = \left(\frac{7}{\sqrt{3}} \right)^2 + \left(\frac{-5}{\sqrt{15}} \right)^2 = 18.$$

Alternatively: $|\vec{v}|^2 = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 18.$ ✓

d) Find an orthogonal basis $\vec{u}_1, \vec{u}_2$ for this inner product.

$$\vec{u}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{(1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right). \qquad \vec{u}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = \frac{(-2,1)}{\sqrt{15}} = \left(-\frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}} \right).$$

e) Write $\vec{v}$ as a linear combination of the orthonormal basis, and verify $|\vec{v}| = \sqrt{c_1^2 + \ldots + c_n^2} = \sqrt{\sum_{i=1}^{n} \langle \vec{v}, \vec{u}_i \rangle^2}$ from the first theorem in this section.

$$\langle \vec{v}, \vec{u}_1 \rangle = \left\langle (3,2), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right\rangle = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \frac{7\sqrt{3}}{3}.$$

$$\langle \vec{v}, \vec{u}_2 \rangle = \left\langle (3,2), \left(-\frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}} \right) \right\rangle = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \end{bmatrix} = -\frac{\sqrt{15}}{3}.$$

$$\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 = \langle \vec{v}, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{v}, \vec{u}_2 \rangle \vec{u}_2 = \frac{7\sqrt{3}}{3} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) - \frac{\sqrt{15}}{3} \left(-\frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}} \right) = \begin{bmatrix} 3 & 2 \end{bmatrix}.$$ ✓

$$|\vec{v}| = \sqrt{\sum_{i=1}^{n} \langle \vec{v}, \vec{u}_i \rangle^2} = \sqrt{\langle \vec{v}, \vec{u}_1 \rangle^2 + \langle \vec{v}, \vec{u}_2 \rangle^2}$$

So, $|\vec{v}| = \sqrt{\left(\frac{7\sqrt{3}}{3} \right)^2 + \left(-\frac{\sqrt{15}}{3} \right)^2} = \sqrt{18}$. ✓ (matches the result in part c)