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3.3 Norms

Every inner product gives rise to a norm that can be used to measure the magnitude or length of the elements of the vector space.

However, not every norm that is used in analysis and applications arises from an inner product.

Definition: A norm on a vector space V assigns a nonnegative real number $|\vec{v}|$ to each vector $\vec{v} \in V$, subject to the following axioms, valid for every $\vec{v}, \vec{w} \in V$ and $c \in \mathbb{R}$:

- Positivity: $|\vec{v}| \geq 0$, with $|\vec{v}| = 0$ iff $\vec{v} = \vec{0}$.
- Homogeneity: $|c\vec{v}| = |c||\vec{v}|$.
- Triangle inequality: $|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$. ("shortest distance between two points is a straight line")

Inner products give rise to a norm:

- Positivity comes from an inner product axiom.
- Homogeneity: $|c\vec{v}| = \sqrt{\langle c\vec{v}, c\vec{v}\rangle} = \sqrt{c^2\langle \vec{v}, \vec{v}\rangle} = |c||\vec{v}|$.
- Triangle Inequality: Theorem 3.9.

Other norms:

Examples: The 1-norm of $\vec{v} = (v_1, \ldots, v_n)$ is: $|\vec{v}|_1 = |v_1| + \ldots + |v_n|$.

Max or ∞-norm: $|\vec{v}|_\infty = \max\{|v_1|, \ldots, |v_n|\}$.

Verification of positivity and homogeneity for these two norms is trivial, and triangle inequality is a result of:

$|a + b| \leq |a| + |b|$ for $a, b \in \mathbb{R}$.

More generally: The p-norm:

$|\vec{v}|_p = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p}$. Norm for all $1 \leq p < \infty$.

Note that:Euclidean norm is the 2-norm and $|\vec{v}|_\infty = \lim_{p \to \infty} |\vec{v}|_p$.

Verification of positivity and homogeneity is not difficult, but showing triangle inequality (*Minkowski's inequality*) is difficult:

$$|\vec{v} + \vec{w}|_p = \sqrt[p]{\sum_{i=1}^{n} |v_i + w_i|^p} \leq \sqrt[p]{\sum_{i=1}^{n} |v_i|^p} + \sqrt[p]{\sum_{i=1}^{n} |w_i|^p} = |\vec{v}|_p + |\vec{w}|_p.$$

Norms of $C^0[a,b]$.

$$|f|_p = \sqrt[p]{\int_a^b |f(x)|^p dx}.$$

$$|f|_1 = \int_a^b |f(x)| dx \qquad |f|_\infty = \max\{|f(x)| : a \le x \le b\}.$$

$|f|_2$ is norm arising from inner product.

- Positivity due to the fact that the only continuous nonnegative function with zero integral is the zero function.
- Homogeneity is easy to establish (try it!).
- Showing Triangle/Minkowski inequality is nontrivial.

Distance

Every norm defines a distance between vector space elements: $d(\vec{v},\vec{w}) = |\vec{v}-\vec{w}|$.

Dot product norm gives the usual Euclidean notion of distance.

Other norms can be useful, but all have the following properties:

a) Symmetry: $d(\vec{v},\vec{w}) = d(\vec{w},\vec{v})$;

b) Positivity: $d(\vec{v},\vec{w}) = 0$ **iff** $\vec{v} = \vec{w}$;

c) Triangle Inequality: $d(\vec{v},\vec{w}) \le d(\vec{v},\vec{z}) + d(\vec{z},\vec{w})$ for all $\vec{z}$.

("shortest distance between two points is a straight line")

Example: Prove that $|\vec{v}| = \sqrt{2v_1^2 - v_1v_2 + 2v_2^2}$ defines a norm on $\mathbb{R}^2$.

We must show positivity, homogeneity, triangle inequality.

Observe that $\sqrt{f} \ge 0$ (and is defined on $\mathbb{R}$) as long as $f \ge 0$. So we need to show that $2v_1^2 + 2v_2^2 \ge v_1v_2$.

Observe that $v_1^2, v_1^2 \ge 0$.

Case: v_1, v_2 opp sign. So, $v_1v_2 \le 0$.

Therefore, $2v_1^2 + 2v_2^2 \ge 0 \ge v_1v_2$.

Case: v_1, v_2 same sign. If $v_1 \ge v_2$, then

$2v_1^2 + 2v_2^2 \ge 2v_1^2 \ge v_1v_1 \ge v_1v_2$.

Similar argument for $v_2 \ge v_1$. So we have shown positivity.

Proving homogeneity: $|c\vec{v}| = \sqrt{2c^2v_1^2 - c^2v_1v_2 + 2c^2v_2^2} = \sqrt{c^2(2v_1^2 - v_1v_2 + 2v_2^2)} = |c||\vec{v}|$.

Triangle Inequality: $|\vec{v}+\vec{w}|^2 = \langle \vec{v}+\vec{w}, \vec{v}+\vec{w}\rangle = \langle \vec{v},\vec{v}\rangle + 2\langle \vec{v},\vec{w}\rangle + \langle \vec{w},\vec{w}\rangle$

$$= |\vec{v}|^2 + 2\langle \vec{v}, \vec{w} \rangle + |\vec{w}|^2 \le |\vec{v}|^2 + 2|\vec{v}||\vec{w}| + |\vec{w}|^2 \qquad \text{(Cauchy-Schwarz)}$$

$$= (|\vec{v}| + |\vec{w}|)^2.$$ Now take square root of both sides. ■

Unit Vectors

Lemma: If $\vec{v} \neq \vec{0}$, then $\vec{u} := \frac{\vec{v}}{|\vec{v}|}$, obtained by dividing $\vec{v}$ by its norm, is a unit vector parallel to $\vec{v}$.

Proof: Making use of the homogeneity property of the norm, and the fact that $|\vec{v}|$ is a scalar,

$$|\vec{u}| = \left| \frac{1}{|\vec{v}|} \vec{v} \right| = \frac{1}{|\vec{v}|} |\vec{v}| = \frac{|\vec{v}|}{|\vec{v}|} = 1.$$ ■

Example: Observe that $\vec{v} = (1, -2)$ has length $|\vec{v}|_2 = \sqrt{5}$ with respect to the standard Euclidean norm.

What is the unit vector pointing in the same direction?

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|_2} = \frac{1}{\sqrt{5}}(1, -2) = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right).$$

On the other hand, for the 1-norm, $|\vec{v}|_1 = 3$. What is the unit vector pointing in the same direction?

$$\tilde{u} = \frac{\vec{v}}{|\vec{v}|_1} = \frac{1}{3}(1, -2) = \left(\frac{1}{3}, -\frac{2}{3} \right).$$

Finally, $|\vec{v}|_\infty = 2$. What is the unit vector pointing in the same direction?

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|_\infty} = \frac{1}{2}(1, -2) = \left(\frac{1}{2}, -1 \right).$$

Example: On $[0, 1]$, $p(x) = x^2 - \frac{1}{2}$ has L^2 norm: $|p|_2 = \sqrt{\int_0^1 \left(x^2 - \frac{1}{2} \right)^2 dx} = \ldots = \sqrt{\frac{7}{60}}$.

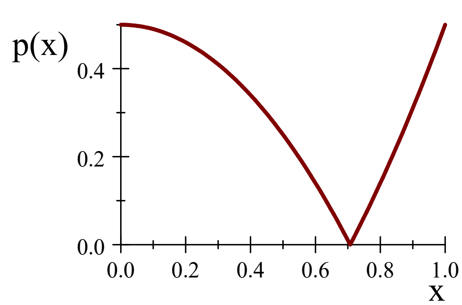
What is the "unit polynomial" which is "parallel" to $p(x)$?

$$u(x) = \frac{p(x)}{|p|} = \sqrt{\frac{60}{7}}\, x^2 - \sqrt{\frac{15}{7}}.$$

How about for the L^∞ norm?

$$|p|_\infty = \max \left\{ \left| x^2 - \frac{1}{2} \right| : 0 \le x \le 1 \right\} = \frac{1}{2}.$$

Therefore: $\tilde{u}(x) = 2p(x) = 2x^2 - 1.$



Unit Sphere

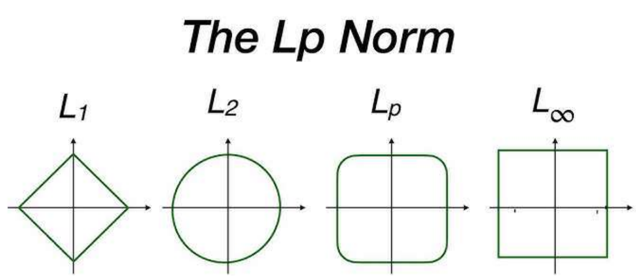
The unit sphere for the given norm is defined as the set of all unit vectors:

$$S_1 := \{\vec{u} : |\vec{u}| = 1\}, \text{ while } S_r := \{\vec{u} : |\vec{u}| = r\}.$$

Unit sphere for Euclidean norm: $S_1 = \left\{\vec{x} : |\vec{x}|^2 = x_1^2 + \ldots + x_n^2 = 1\right\}$.

Unit sphere for ∞-norm is surface of unit cube: $S_1 = \left\{\vec{x} : \begin{array}{l} |x_i| \le 1,\ i = 1,\ldots,n, \text{ and either} \\ x_1 = \pm 1 \text{ or } x_2 = \pm 1 \text{ or } \ldots \text{ or } x_n = \pm 1. \end{array}\right\}$.

For the 1-norm: $S_1 = \{\vec{x} : |x_1| + \ldots + |x_n| = 1\}$, The unit diamond in 2D.



$$|\vec{v}|_p = 1$$

Equivalence of Norms

$\exists$ many different norms in a finite-dimensional vector space. They are all more or less "equivalent."

"Equivalence" does *not* mean that they have same values (as seen above), but for many purposes, may be used interchangeably.

And we can simplify analysis of a problem by choosing a suitable norm.

Equivalence of norms on finite-dimensional vector spaces:

Theorem: Let $|\cdot|_a$ & $|\cdot|_b$ be any two norms on $\mathbb{R}^n$. Then there exist $0 < c^* \le C^*$ such that:

$$c^* |\vec{v}|_a \le |\vec{v}|_b \le C^* |\vec{v}|_a \text{ for every } \vec{v} \in \mathbb{R}^n. \qquad (*)$$

Proof: We just sketch basic idea (rigorous proof given in real analysis course).

Note that each norm defines a continuous $f(\vec{v}) = |\vec{v}|$ on $\mathbb{R}^n$.

Let $S_a = \left\{|\vec{u}|_a = 1\right\}$ denote unit sphere of $|\cdot|_a$.

Recall: every continuous function defined on a compact set achieves both a maximum and minimum value.

Thus, restricting the b-norm function $|\vec{v}|_b$ to the unit sphere S_a of the a-norm, we can set:

$$c^* = \min\left\{|\vec{u}|_b : \vec{u} \in S_a\right\}, \quad C^* = \max\left\{|\vec{u}|_b : \vec{u} \in S_a\right\}. \qquad (**)$$

Moreover, $0 < c^* \leq C^* < \infty$, with equality holding **iff** the norms are the same.

The maximum and minimum $(**)$ will serve as the constants in the desired inequalities.

Indeed, by definition: $c^* \leq |\vec{u}|_b \leq C^*$ when $|\vec{u}|_a = 1$, $\qquad (***)$

which proves that $(*)$ is valid for all unit vectors $\vec{v} \in S_a$.

To prove the inequalities in general, assume $\vec{v} \neq 0$ (the case $\vec{v} = \vec{0}$ is trivial).

A previous lemma says $\vec{u} = \frac{\vec{v}}{|\vec{v}|_a} \in S_a$ is unit vector in the a-norm: $|\vec{u}|_a = 1$.

Moreover, by the homogeneity property of the norm, $|\vec{u}|_b = \left|\frac{\vec{v}}{|\vec{v}|_a}\right|_b = \frac{1}{|\vec{v}|_a}|\vec{v}|_b = \frac{|\vec{v}|_b}{|\vec{v}|_a}$.

Substituting this into $(***)$ and clearing denominators completes the proof. ■

Equivalence of the ∞ and 2 Norms

Consider $|\cdot|_2$ and $|\cdot|_\infty$. According to $(**)$, bounding constants are found by minimizing and

maximizing $|\vec{u}|_\infty = \max\{|u_1|, \ldots, |u_n|\}$ over all unit vectors $|\vec{u}|_2 = 1$ on the (round) unit sphere.

The maximal value is achieved at the poles $\pm\vec{e}_k$, with $C^* := |\pm\vec{e}_k|_\infty = 1$.

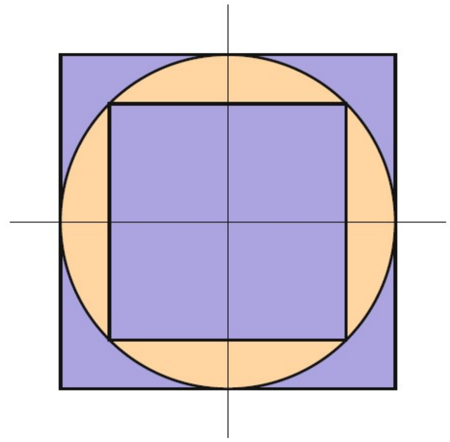
The minimal value is obtained at the points $\left(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}}\right)$, whereby $c^* := \frac{1}{\sqrt{n}}$.

Therefore, $\frac{1}{\sqrt{n}}|\vec{v}|_2 \leq |\vec{v}|_\infty \leq |\vec{v}|_2$. $\qquad (****)$

Interpretation: Suppose $\vec{v}$ is lying on unit sphere in $|\cdot|_2$, so $\left|\vec{v}\right|_2 = 1$.

Then $(***)$ tells us that its ∞-norm is bounded above & below as $\frac{1}{\sqrt{n}} \leq \left|\vec{v}\right|_\infty \leq 1$.

So Euclidean unit sphere sits inside ∞-norm unit sphere and outside ∞-norm sphere of radius $\frac{1}{\sqrt{n}}$.



Equivalence of the ∞ and 2 Norms.