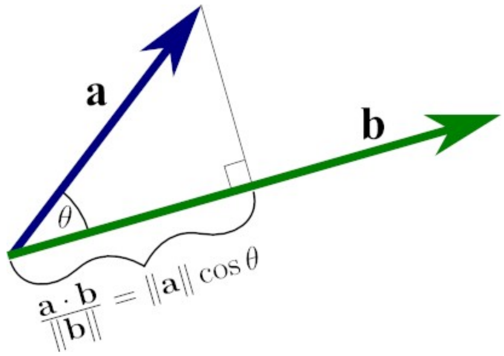
3.2 Cauchy-Schwarz and Triangle Inequalities

There are two basic inequalities that are valid for *any* inner product space: **Cauchy-Schwarz** and the **Triangle** Inequality.

The Cauchy-Schwarz Inequality



Recall: Dot product of any two $\vec{v},\vec{w} \in \mathbb{R}^n$ is: $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta$.

And since $|\cos\theta| \leq 1$, we have $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

More generally:

Theorem: Every inner product satisfies the Cauchy Schwartz inequality: $|\langle\vec{v},\vec{w}\rangle| \leq |\vec{v}||\vec{w}|$, for all $\vec{v},\vec{w} \in V$.

Here, the meaning of $|\cdot|$ is contextual. If $\cdot$ is a vector, then $|\cdot|$ means the associated norm.
If $\cdot$ is a scalar, then $|\cdot|$ means the absolute value.

Proof: The case when $\vec{w} = \vec{0}$ is trivial, since both sides of the inequality are equal to zero.

Thus, we concentrate on the case when $\vec{w} \neq \vec{0}$.

Let $t \in \mathbb{R}$. Using the three inner product axioms, we have:

$0 \leq |\vec{v} + t\vec{w}|^2$ (positivity)

$= \langle\vec{v} + t\vec{w},\ \vec{v} + t\vec{w}\rangle = \langle\vec{v},\vec{v}\rangle + 2t\langle\vec{v},\vec{w}\rangle + t^2\langle\vec{w},\vec{w}\rangle$ (bilinearity & symmetry)

$= |\vec{v}|^2 + 2t\langle\vec{v},\vec{w}\rangle + t^2|\vec{w}|^2,$ $(*)$

with inequality holding **iff** $\vec{v} = -t\vec{w}$, which requires $\vec{v},\vec{w}$ to be parallel.

Now, we fix $\vec{v},\vec{w}$, and consider $(*)$ as a quadratic function of t.

So, $0 \leq p(t) := at^2 + 2bt + c$, where $a = |\vec{w}|^2$, $b = \langle\vec{v},\vec{w}\rangle$, $c = |\vec{v}|^2$.

To get the most out of the fact that $p(t) \geq 0$, let us look at where it assumes its minimum, which occurs when its derivative is 0:

$$p'(t) = 2at + 2b = 0, \quad \text{and so} \quad t = -\frac{b}{a} = -\frac{\langle \vec{v}, \vec{w} \rangle}{|\vec{w}|^2}.$$

Substituting this particular value of t into $(*)$, we obtain: $0 \leq |\vec{v}|^2 - 2\frac{\langle \vec{v}, \vec{w} \rangle^2}{|\vec{w}|^2} + \frac{\langle \vec{v}, \vec{w} \rangle^2}{|\vec{w}|^2} = |\vec{v}|^2 - \frac{\langle \vec{v}, \vec{w} \rangle^2}{|\vec{w}|^2}$.

Rearranging this last inequality, we conclude that: $\frac{\langle \vec{v}, \vec{w} \rangle^2}{|\vec{w}|^2} \leq |\vec{v}|^2$, or $\langle \vec{v}, \vec{w} \rangle^2 \leq |\vec{v}|^2 |\vec{w}|^2$. $\quad (**)$

Also, as noted above, equality holds **iff** $\vec{v} \parallel \vec{w}$.

Equality also holds when $\vec{w} = 0$, which is of course parallel to every vector $\vec{v}$.

Taking the (positive) square root of $(**)$ completes the proof. ■

Recall the dot product in $\mathbb{R}^n$ ($\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$), can be used to define the angle θ between $\vec{v}, \vec{w} \in V$.

Similarly, we can define the "angle" between more general vector space element $\vec{v}, \vec{w} \in V$ with: $\cos\theta = \frac{\langle \vec{v}, \vec{w} \rangle}{|\vec{v}||\vec{w}|}$.

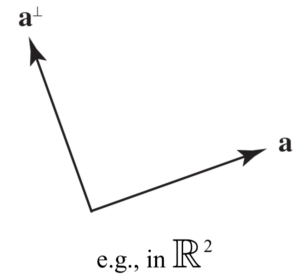
For Instance: With L^2 inner product on $[0, 1]$, the "angle" θ between polynomial $p(x) = x$ and $q(x) = x^2$ is given by:

$$\cos\theta = \frac{\langle x, x^2 \rangle}{|x||x^2|}$$

$$= \frac{\int_0^1 x^3\,dx}{\sqrt{\int_0^1 x^2\,dx}\sqrt{\int_0^1 x^4\,dx}}$$

$$= \frac{\frac{1}{4}}{\sqrt{\frac{1}{3}}\sqrt{\frac{1}{5}}} = \sqrt{\frac{15}{16}}, \text{ so that } \theta = \cos^{-1}\sqrt{\frac{15}{16}} = 0.25268\ldots \text{ radians.}$$

Orthogonal Vectors



Recall in $\mathbb{R}^n$, vectors $\vec{v}, \vec{w}$ are orthogonal (perpendicular) if their dot product (inner product) is zero.

Like angle, we can generalize this to any inner product space:

Definition: Two elements $\vec{v}, \vec{w} \in V$ of an inner product space V are called orthogonal if their inner product vanishes: $\langle \vec{v}, \vec{w} \rangle = 0$.

In particular, $\vec{0}$ is orthogonal to every other element $\vec{v}$ in an inner product space.

For Instance: $\vec{v} = (1,2)$ and $\vec{w} = (6,-3)$ are orthogonal with respect to the Euclidean dot product in $\mathbb{R}^2$.

However, if we have the weighted inner product: $\langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + 5v_2w_2$,

then observe: $\langle \vec{v}, \vec{w} \rangle = (2 \cdot 1 \cdot 6) + (5 \cdot 2 \cdot (-3)) = -18 \neq 0$.

Therefore, $\vec{v}, \vec{w}$ are not orthogonal in this weighted inner product.

Example: Show polynomials $p(x) = x$ and $q(x) = x^2 - \frac{1}{2}$ are orthogonal with to respect to inner product:

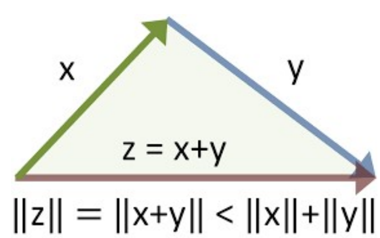
$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$ on $[0,1]$.

$$\left\langle x, x^2 - \tfrac{1}{2} \right\rangle = \int_0^1 x\left(x^2 - \tfrac{1}{2}\right)dx = \int_0^1 \left(x^3 - \tfrac{1}{2}x\right)dx = 0.$$

But if we switch the interval to $[0,2]$, in this new inner product space, they are **not** orthogonal:

$$\left\langle x, x^2 - \tfrac{1}{2} \right\rangle = \ldots = \int_0^2 \left(x^3 - \tfrac{1}{2}x\right)dx = 3.$$

The Triangle Inequality



Theorem: The norm associated with an inner product satisfies the **Triangle Inequality**: $\left|\vec{v} + \vec{w}\right| \leq \left|\vec{v}\right| + \left|\vec{w}\right|$ for all $\vec{v}, \vec{w} \in V$.

Equality holds **iff** $\vec{v}, \vec{w}$ are parallel vectors.

Proof: $|\vec{v}+\vec{w}|^2 = \langle \vec{v}+\vec{w},\ \vec{v}+\vec{w}\rangle = |\vec{v}|^2 + 2\langle \vec{v},\vec{w}\rangle + |\vec{w}|^2$ (bilinearity & symmetry)

$\leq |\vec{v}|^2 + 2|\vec{v}||\vec{w}| + |\vec{w}|^2$ (Cauchy Schwartz, see exercise 3.2.11)

$= (|\vec{v}| + |\vec{w}|)^2.$

Take square roots of both sides. Since both expressions are positive, this completes the proof. ■

Example: Verify triangle inequality with: $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, and Euclidean norm.

The vectors sum to $\vec{v}+\vec{w} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

Their Euclidean norms are $|\vec{v}| = \sqrt{6}$ and $|\vec{w}| = \sqrt{13}$, while $|\vec{v}+\vec{w}| = \sqrt{17}$.

Triangle Inequality says: $4.1231 \approx \sqrt{17} \leq \sqrt{6} + \sqrt{13} \approx 6.055.$ True!

Example: Verify triangle inequality with L^2 norm on the interval $[0,1]$ with functions: $f(x) = x - 1$ and $g(x) = x^2 + 1$.

$|f| = \sqrt{\int_0^1 (x-1)^2 dx} = \sqrt{\tfrac{1}{3}}$, $|g| = \sqrt{\int_0^1 (x^2+1)^2 dx} = \sqrt{\tfrac{28}{15}}$,

$|f+g| = \sqrt{\int_0^1 (x^2+x)^2 dx} = \sqrt{\tfrac{31}{30}}$.

Triangle Inequality says: $1.0165 \approx \sqrt{\tfrac{31}{30}} \leq \sqrt{\tfrac{1}{3}} + \sqrt{\tfrac{28}{15}} \approx 1.9436.$