Applied Linear Algebra

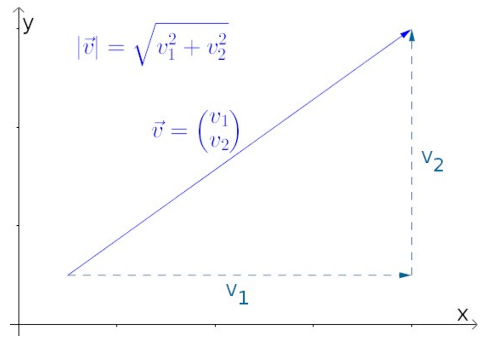
Instructor: Jodin Morey moreyjc@umn.edu

3.1 Inner Products

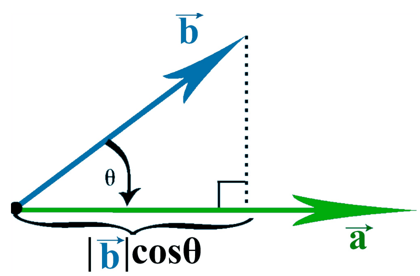
Dot Product (or scalar product): $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \ldots + a_n b_n = \sum_{i=1}^n a_i b_i$.

In particular: $\vec{x} \cdot \vec{x} = x_1^2 + \ldots x_n^2$.

And so, the Pythagorean Thm comes from the square of length. *Euclidean Norm*: $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + \ldots x_n^2}$.



Alternatively (and equivalently): If θ is angle between $\vec{a}$, $\vec{b}$ where $0 \leq \theta \leq \pi$, then $\vec{a} \cdot \vec{b} = \langle \vec{a}, \vec{b} \rangle = |\vec{a}| |\vec{b}| \cos\theta$. If $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a} \cdot \vec{b} = 0$.



$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta$$

Observation: If θ is angle between (nonzero) vectors $\vec{a}, \vec{b}$, then: $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

More generally:

Definition: An inner product on the real vector space V is a pairing that takes two vectors $\vec{v}, \vec{w} \in V$; produces a real number $\langle \vec{v}, \vec{w} \rangle \in \mathbb{R}$; and satisfies the following three axioms for all $\vec{u}, \vec{v}, \vec{w} \in V$, and scalars $c, d \in \mathbb{R}$.

- Bilinearity: $\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = c\langle \vec{u}, \vec{w} \rangle + d\langle \vec{v}, \vec{w} \rangle$, and $\langle \vec{u}, c\vec{v} + d\vec{w} \rangle = c\langle \vec{u}, \vec{v} \rangle + d\langle \vec{u}, \vec{w} \rangle$.

- Symmetry: $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$.

- Positivity: $\langle \vec{v}, \vec{v} \rangle > 0$ whenever $\vec{v} \neq \vec{0}$, while $\langle \vec{0}, \vec{0} \rangle = 0$.

Given an inner product, the associated norm of $\vec{v} \in V$ is the positive square root of inner product of the vector with itself:

$$|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

In Euclidean geometry, θ between two vectors can be formalized into the abstract concept of an inner product (defined below).

Inner products and norms lie at the heart of linear (and nonlinear) analysis, in both finite and infinite dimensional vector spaces.

Definition: A vector space equipped with an inner product and its associated norm is known as an **inner product space**.

There are inner products on $\mathbb{R}^n$ other than the dot product.

Weighted Inner Product

$\langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + 5v_2w_2$. One can verify this satisfies the axioms above.

The associated weighted norm: $|\vec{v}| = \sqrt{2v_1^2 + 5v_2^2}$ defines an alternative "non-Pythagorean" notion of length of vectors and distance between points.

Example: Determine whether the following formulas for $\langle \vec{v}, \vec{w} \rangle$ define inner products on $\mathbb{R}^2$:

$\langle \vec{v}, \vec{w} \rangle = (v_1 + v_2)(w_1 + w_2)$, and $\langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + (v_1 - v_2)(w_1 - w_2)$.

Bilinearity: $\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = \left(cu_1 + dv_1 + cu_2 + dv_2,\ w_1 + w_2\right)$.

Also: $c\langle \vec{u}, \vec{w} \rangle + d\langle \vec{v}, \vec{w} \rangle = (cu_1 + cu_2, w_1 + w_2) + (dv_1 + dv_2, w_1 + w_2) = \left(cu_1 + dv_1 + cu_2 + dv_2,\ 2w_1 + 2w_2\right)$, and therefore it does not pass bilinearity.

Symmetry: $\langle \vec{v}, \vec{w} \rangle = (v_1 + v_2)(w_1 + w_2) = (w_1 + w_2)(v_1 + v_2) = \langle \vec{w}, \vec{v} \rangle$. ✓

Positivity: $\langle \vec{v}, \vec{v} \rangle = (v_1 + v_2)(v_1 + v_2) = 2(v_1 + v_2)$. Observe that $\langle \vec{v}, \vec{v} \rangle$ is zero if $\vec{v} = 0$.

However, if $v_1 = -v_2 \neq 0$, then $\langle \vec{v}, \vec{v} \rangle = 0$. Therefore it does not pass positivity.

Example: $\langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + (v_1 - v_2)(w_1 - w_2)$

Bilinearity: $\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = 2w_1(cu_1 + dv_1) + ((cu_1 + dv_1) - (cu_2 + dv_2))(w_1 - w_2)$

$$\langle c\vec{u}, \vec{w} \rangle + \langle d\vec{v}, \vec{w} \rangle = 2cu_1w_1 + c(u_1 - u_2)(w_1 - w_2) + 2dv_1w_1 + d(v_1 - v_2)(w_1 - w_2)$$
$$= 2w_1(cu_1 + dv_1) + ((cu_1 + dv_1) - (cu_2 + dv_2))(w_1 - w_2) \quad ✓$$

Symmetry: $\langle \vec{v}, \vec{w} \rangle = 2v_1w_1 + (v_1 - v_2)(w_1 - w_2) = 2w_1v_1 + (w_1 - w_2)(v_1 - v_2) = \langle \vec{v}, \vec{w} \rangle$ ✓

Positivity: $\langle \vec{v}, \vec{v} \rangle = 2v_1^2 + 2(v_1 - v_2)$. Observe that $\langle \vec{v}, \vec{v} \rangle$ is zero if $\vec{v} = 0$.

But also observe that when $v_1 = 1$, we have $2 \cdot 1 + 2(1 - v_2) = 0$ when $v_2 = 2$. Therefore it does not pass positivity.

Inner Products on Function Spaces

Let $[a,b] \subseteq \mathbb{R}$ be a bounded, closed interval. Function space $C^0[a,b]$ consists of all continuous, scalar functions f defined on $[a,b]$.

Claim (proved below): an inner product of $C^0[a,b]$ can be defined by the integral of the product of two continuous functions f, g:

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx.$$

The associated norm is $|f| = \sqrt{\int_a^b f(x)^2 \, dx}$, and is known as the L^2 **norm** of the function f over the interval $[a,b]$.

The L^2 inner product and norm of the functions can be viewed as the infinite-dimensional function space versions of the dot product and Euclidean norm of the vectors in $\mathbb{R}^2$.

Example: If $[a,b] = \left[0, \frac{\pi}{2}\right]$, then L^2 inner product between $f(x) = \sin x$ and $g(x) = \cos x$ is equal to ...

$$\langle \sin x, \cos x \rangle = \int_0^{\frac{\pi}{2}} \sin x \cos x \, dx = \frac{1}{2} \sin^2 x \Big|_{x=0}^{\frac{\pi}{2}} = \frac{1}{2}. \qquad \text{(u-sub } u = \sin x\text{)}$$

Norm of $\sin x$ is $|\sin x| =$??

$$= \sqrt{\int_0^{\frac{\pi}{2}} (\sin x)^2 \, dx} = \ldots = \sqrt{\frac{\pi}{4}}.$$

Warning 1: $|1| = \sqrt{\int_0^{\frac{\pi}{2}} 1^2 \, dx} = \sqrt{\frac{\pi}{2}} \neq 1.$

Warning 2: Norm depends on the interval: For example, on $[0, \pi]$:

$|1| = \sqrt{\int_0^{\pi} 1^2 \, dx} = \sqrt{\pi}.$ $\quad L^2$ inner product & norm depend upon the interval of the function space!

Now we Prove the above claim ($\langle f, g \rangle = \int_a^b f(x) g(x) dx$ is inner product):

First, is $\langle f, g \rangle = \int_a^b f(x) g(x) dx$ even **well defined**?

Notice that the product of two continuous functions is also continuous, and the integral over a bounded interval is defined and finite. ✓

Symmetric Requirement: Need: $\langle f, g \rangle = \langle g, f \rangle$.

$$\langle f, g \rangle = \int_a^b fg \, dx = \int_a^b gf \, dx = \langle g, f \rangle. \qquad ✓$$

First Bilinearity Axiom: Need: $\langle cf+dg, h\rangle = c\langle f,h\rangle + d\langle g,h\rangle$. Follows from observing that

$$\int_a^b [cf(x)+dg(x)]h(x)\,dx = c\int_a^b f(x)h(x)\,dx + d\int_a^b g(x)h(x)\,dx. \qquad \checkmark$$

Second Bilinearity Axiom: similarly proved. ✓

Positivity requires that $|f|^2 = \langle f, f \rangle = \int_a^b f(x)^2 dx \geq 0$.

But, observe that $f(x)^2 \geq 0$.

And since $f(x)^2$ continuous, nonnegative, $|f|^2 = \int_a^b f(x)^2 dx = 0$ (norm is zero) **iff**

$f(x) \equiv 0$ (the zero function). ✓ ■

❗ Warning: Be careful when extending L^2 inner product to other spaces of functions.

Nonzero discontinuous functions with zero "L^2 norm."

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{satisfies } |f|^2 = \int_{-1}^1 f(x)^2 dx = 0.$$

L^2 is but one example of an inner product on function spaces.

Weighted inner product space on the space $C^0[a,b]$. For a continuous, positive, scalar function $w(x) > 0$.

Weighted Inner Product and Norm: $\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\,dx, \qquad |f| = \sqrt{\int_a^b f(x)^2 w(x)\,dx}\,.$

Weighted inner products used in statistics and data analysis.

Example: For $f(x) = \cos(2\pi x)$, and $g(x) = \sin(2\pi x)$ in $C^0[0,1]$, find their L^2 inner product $\langle f,g\rangle$ and their L^2 norms $|f|, |g|$.

$$\langle f,g\rangle = \int_0^1 f(x)g(x)dx = \int_0^1 \cos(2\pi x)\sin(2\pi x)dx$$

$$= \tfrac{1}{2}\int_0^1 \sin(4\pi x)dx = \tfrac{1}{8\pi}[-\cos(4\pi x)]_0^1 \qquad \text{(double angle trig ident.)}$$

$$= \tfrac{1}{8\pi}(-\cos(4\pi) + \cos(0)) = \tfrac{1}{8\pi}(0) = 0.$$

$$|f| = \sqrt{\int_a^b f(x)^2\,dx} = \sqrt{\int_0^1 \cos^2(2\pi x)\,dx}$$

$$= \sqrt{\int_0^1 \tfrac{1+\cos(4\pi x)}{2}\,dx} \qquad \text{(double angle trig ident.)}$$

$$= \sqrt{\left[\tfrac{1}{2}x + \tfrac{1}{8\pi}\sin(4\pi x)\right]_0^1} = \sqrt{\left(\tfrac{1}{2}+0\right) - (0+0)} = \tfrac{1}{\sqrt{2}}.$$

$$|g| = \sqrt{\int_a^b g(x)^2\,dx} = \sqrt{\int_0^1 \sin^2(2\pi x)\,dx}$$

$$= \sqrt{\int_0^1 \tfrac{1-\cos(4\pi x)}{2}\,dx} = \sqrt{\left[\tfrac{1}{2}x - \tfrac{1}{8\pi}\sin(4\pi x)\right]_0^1} = \sqrt{\left(\tfrac{1}{2}+0\right) - (0+0)} = \tfrac{1}{\sqrt{2}}. \qquad \text{(double angle trig ident.)}$$

Example: For $f(x) = (x+1)^2$, and $g(x) = \frac{1}{x+1}$ in $C^0[0,1]$, find their L^2 inner product $\langle f,g\rangle$ and their L^2 norms $|f|, |g|$.

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx = \int_0^1 \tfrac{(x+1)^2}{x+1}\,dx$$

$$= \int_0^1 (x+1)\,dx = \left[\tfrac{1}{2}x^2 + x\right]_0^1 = \tfrac{3}{2}.$$

$$|f| = \sqrt{\int_a^b f(x)^2\,dx} = \sqrt{\int_0^1 (x+1)^4\,dx}$$

$$= \sqrt{\left[\tfrac{1}{5}(x+1)^5\right]_0^1} = \sqrt{\tfrac{2^5}{5} - \tfrac{1}{5}} = \sqrt{\tfrac{31}{5}}.$$

$$|g| = \sqrt{\int_a^b g(x)^2\,dx} = \sqrt{\int_0^1 (x+1)^{-2}\,dx}$$

$$= \sqrt{\left[-(x+1)^{-1}\right]_0^1} = \sqrt{-\tfrac{1}{2}+1} = \tfrac{1}{\sqrt{2}}.$$