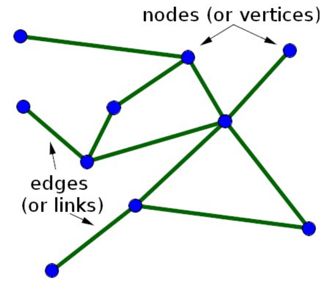
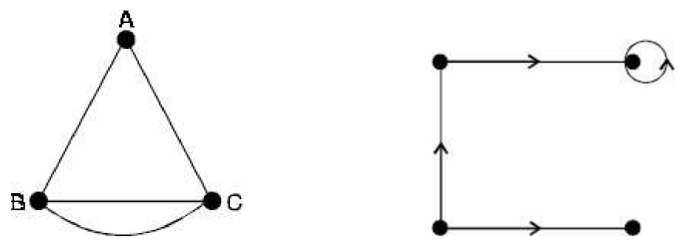
2.6 Graphs and Digraphs



Definitions:

Graph: finite # of points, called *vertices*, and finitely many lines or curves connecting them, called *edges*.

Each edge connects exactly two vertices, which are its *endpoints*. To avoid technicalities, we will always assume the graph is *simple*, i.e., that every edge connects two distinct vertices, so no edge forms a *loop* that connects a vertex to itself, and, moreover, two distinct vertices are connected by at most one edge.



Non-Simple Graphs

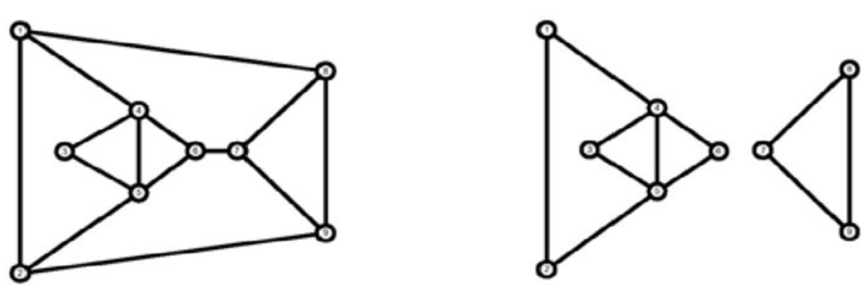
Graphs encode **topology** (interconnectedness) of the system, but not its **geometry** (not the length of edges, angles, etc.).

Two graphs are the same if there's a **one-to-one correspondence** between their edges and their vertices, so that matched edges connect matched vertices.

Definitions:

A *path* in a graph is an ordered list of distinct edges $e_1, \ldots, e_k$ connecting vertices $v_1, \ldots, v_{k+1}$ so that edge e_i connects vertex v_i to v_{i+1}. While an edge cannot be repeated in a path, a vertex may be.

A graph is *connected* if you can get from any vertex to any other vertex by a path.

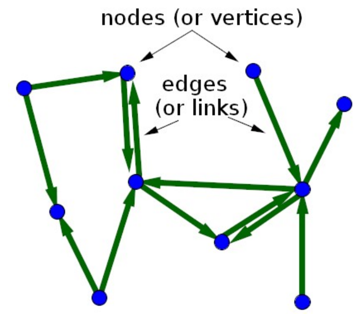


Connected Disconnected

A *circuit* (closed path) is a path that ends up where it began, i.e., $v_{k+1} = v_1$.

Two circuits that go around the edges in the same order, but with two different starting points, are the **same circuit**.

A graph with directed edges is known as a *directed graph*, or *digraph* for short.

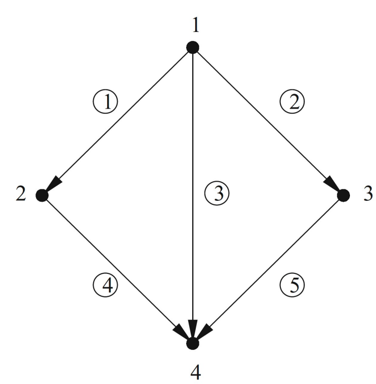


Directed Graph

Orientation of an edge will be fixed by specifying the vertex the edge starts at, and the vertex it ends at.

Once we assign a *direction*, a current along that edge is positive if it moves in the same direction.

Consider a digraph D consisting of n vertices connected by m edges. The *incidence matrix* associated with D is an $m \times n$ matrix $\mathbf{A}$ whose rows are indexed by the edges and whose columns are indexed by the vertices.



Digraph D

Example: The graph above consists of five edges joined at four different vertices:

Its 5×4 incidence matrix is: $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Thus the first row of $\mathbf{A}$ tells us that the first edge starts at vertex 1 and ends at vertex 2, etc.

Incidence Matrix Subspaces

The incidence matrix of a digraph encodes geometric information.

Kernel

Proposition: If $\mathbf{A}$ is the incidence matrix for a connected digraph, then $\ker \mathbf{A}$ is one-dimensional, with basis $\vec{z} = (1, 1, \ldots, 1)$.

Proof: If edge k connected vertex i to vertex j, then the k^{th} equation in $\mathbf{A}\vec{z} = \vec{0}$ is $z_i - z_j = 0$, or, equivalently, $z_i = z_j$.

The same equality holds, by a simple induction, if the vertices i and j are connected by a **path**.

Therefore, if D is **connected**, then all the entries of $\vec{z}$ are equal, and $\ker \mathbf{A} = \mathbf{span}\{(\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1})\}$. ■

Remark: In general, dim ker $\mathbf{A}$ equals the number of connected components in the digraph D.

Applying the fundamental theorem of linear algebra, we find:

Corollary: If $\mathbf{A}$ is the incidence matrix for a connected digraph with n vertices, then $rank\,\mathbf{A} = n - 1$.

Cokernel

Looking at our previous example, the cokernel would come from calculating the kernel of the transposed matrix:

$$\mathbf{A}^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}.$$

Solving this we find the kernel spanned by $\vec{y}_1 = (1, 0, -1, 1, 0)$ and $\vec{y}_2 = (0, 1, -1, 0, 1)$.

Each vector represents a circuit in the digraph. Recall entries are indexed by edges.
Nonzero entry indicates the direction to traverse the corresponding edge.

For example, $\vec{y}_1$ corresponds to the circuit that starts out along edge 1, then goes along edge 4,
and finishes by going along edge 3 in the reverse direction.

If y_1 and y_2 are linearly independent vectors, we say the two circuits are "*independent*".

Theorem: Each circuit in a digraph D is represented by a vector in the cokernel of its incidence matrix $\mathbf{A}$,
whose entries are $+1$ if the edge is traversed in the correct direction, -1 if in the opposite direction,
and 0 if the edge is not in the circuit. The dimension of the cokernel of $\mathbf{A}$ equals the number of independent circuits in D.

Euler's Formula for Graphs: Suppose D is a connected digraph with m edges and n vertices and $\mathbf{A}$ its $m \times n$ incidence matrix.

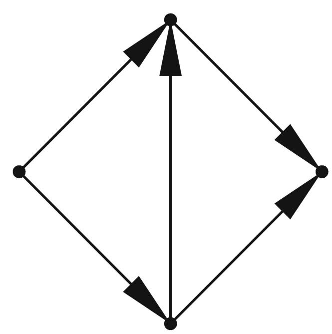
A previous corollary implies that $\mathbf{A}$ has rank $r = n - 1 = n - \dim \ker \mathbf{A}$.

On the other hand, the above theorem tells us that $\ell = \dim co\ker \mathbf{A}$ equals the number of independent circuits in D.

Also, the fundamental theorem of linear algebra says that $r = m - \ell$.

Equating these two formulas for the rank, we obtain: $r = n - 1 = m - \ell$, or $n + \ell = m + 1$. Therefore...

Theorem: If G is a connected graph, then # vertices + # independent circuits = # edges +1.



Example: Write out the incidence matrix $\mathbf{A}$ of the digraph above.

Find a basis of the cokernel of the incidence matrix.

Determine the dimension of the cokernel.

What does this tell you about the number of independent circuits in the digraph?

We label the vertices by starting out with 1 as the top vertex, and moving counterclockwise.

The upper left edge is 1, the upper right is 3, the center edge is 2, the lower left is 4, with the lower right being 5. Therefore:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

$$\mathbf{A}^T = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_5 = c_5,\ c_4 = c_4,\ c_3 = -c_5,\ c_2 = c_4 - c_5,\ c_1 = -(c_4 - c_5) - c_5 = -c_4.$

$(-c_4, c_4 - c_5, -c_5, c_4, c_5) = c_4(-1,1,0,1,0) + c_5(0,-1,-1,0,1),$

$\ker \mathbf{A}^T = co\ker \mathbf{A} = span\{(-1,1,0,1,0),(0,-1,-1,0,1)\}.$

$\dim(co\ker \mathbf{A}) = 2.$

Two independent circuits in the digraph.