Instructor: Jodin Morey moreyjc@umn.edu

2.5 The Fundamental Matrix Subspaces

Definition: The **image** (or **column space** or **range**) of an $m \times n$ matrix $\mathbf{A}$ is the subspace: $img\,\mathbf{A} \subset \mathbb{R}^m$ spanned by its columns.

$$img\,\mathbf{A} = span\{cols\,\mathbf{A}\} \subseteq \mathbb{R}^m$$

$$= \left\{b \in \mathbb{R}^m \,|\, b = a_1c_1 + \ldots + a_nc_n \text{ where } a_i \in \mathbb{R}, \text{ and } c_i \text{ are columns of } \mathbf{A}\right\}$$

$$= \left\{b \in \mathbb{R}^m \mid \mathbf{A}\vec{c} = \vec{b} \text{ has a solution}\right\}.$$

Definition: The *kernel of A* is the subspace $\ker \mathbf{A} \subset \mathbb{R}^n$ consisting of all vectors that are *annihilated* by $\mathbf{A}$,

so $\ker \mathbf{A} = \left\{z \in \mathbb{R}^n \,|\, \mathbf{A}\vec{z} = \vec{0}\right\} \subset \mathbb{R}^n$.

Example: $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$ $(m = 2,\ n = 3)$

$$img\,\mathbf{A} = span\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}\right\}.$$

Note: This is not a basis, as columns are dependent:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -4 \end{bmatrix}$$ (in fact, all columns parallel)

Row reduced: $\rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{U}$.

So: $img\,\mathbf{A}$ is one-dimensional, with basis $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

In general: A basis for $img\,\mathbf{A}$ is given by the columns of $\mathbf{A}$ with pivots.

So: $\dim img\,\mathbf{A}$ = # cols with pivots = $rank\,\mathbf{A}$.

Caution: The columns of $\mathbf{U}$ with pivots are not a basis for $img\,\mathbf{A}$, and usually don't belong to $img\,\mathbf{A}$.

For example: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in the previous example.

Remark: If $\mathbf{A}$ is an $m \times n$ matrix: $\ker \mathbf{A}$ lives in $\mathbb{R}^n$ and $img\,\mathbf{A}$ lives in $\mathbb{R}^m$.

Observe: # cols with pivots is dim $img\,\mathbf{A}$.

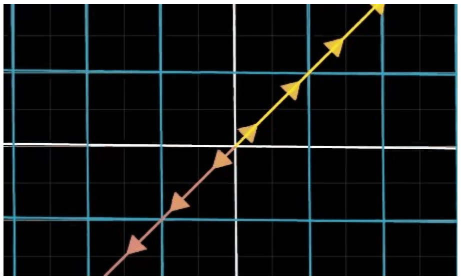
cols without pivots is dim ker $\mathbf{A}$.

Conclude:

Rank-Nullity Theorem: Let $\mathbf{A}$ be an $m \times n$ matrix. Then $n = \dim img\,\mathbf{A} + \dim \ker \mathbf{A} = rank\,\mathbf{A} + nullity\,\mathbf{A}$.

Intuitively: the input is m-dimensional.

Each of m dimensions is either killed off (goes to $\vec{0}$), or survives to the image.



[see animation in class]

Example: $\mathbf{A}^{5\times7}$ of rank 3 has $\dim \ker \mathbf{A} =$??

$= 4.$

$\mathbf{A}^{6\times8}$ with $\dim \ker \mathbf{A} = 2$ has *rank* 6.

Example: $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -9 \\ -2 & 4 & -6 \\ 3 & 0 & -1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 6 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

So first and second cols have pivots.

Thus, $\begin{bmatrix} 1 \\ -3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 4 \\ 0 \end{bmatrix}$ is a basis for $img\,\mathbf{A}$, which is two-dimensional subspace of $\mathbb{R}^4$.

One-dimensional kernel of $\mathbb{R}^3$.

Finding General Solutions

Fix $\mathbf{A}$ and $\vec{b}$, and suppose there are two solutions: $\mathbf{A}\vec{x} = \vec{b}$ and $\mathbf{A}\vec{y} = \vec{b}$.

Then their difference $\vec{z} = \vec{x} - \vec{y}$ satisfies: $\mathbf{A}\vec{z} = \mathbf{A}(\vec{x} - \vec{y}) = \mathbf{A}\vec{x} - \mathbf{A}\vec{y} = \vec{b} - \vec{b} = \vec{0}$.

Conclude: The difference of two solutions $\vec{z}$ is in the kernel of $\mathbf{A}$.

And, every solution $\vec{x}$ can be written as $\vec{x} = \vec{y} + \vec{z}$, for any given solution $\vec{y}$ and an element $\vec{z}$ of the kernel.

Thus, if we know:

- $\ker \mathbf{A}$, and
- a single solution $\vec{x}^*$ to $\mathbf{A}\vec{x} = \vec{b}$,

then we can find all solutions (!!).

Theorem: Suppose x^* is a solution to $\mathbf{A}\vec{x} = \vec{b}$. Then, all other solutions are of the form: $x^* + z$, where $z \in \ker \mathbf{A}$.

Example: Let $\mathbf{A} := \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix}$ and $\vec{b} := \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Row reduce $\left[\mathbf{A}|\vec{b}\right]$ to get: $\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & -2 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$.

Note c_3 is free. So, if we set $c_3 = 0$, we find $c_1 = 3$ and $c_2 = 1$.

So, $x^* = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is one solution.

Next, find kernel. General solution to $\mathbf{A}\vec{c} = \vec{0}$.

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

So, $\ker \mathbf{A} = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Thus, *all* solutions to $\mathbf{A}\vec{x} = \vec{b}$ are of the form $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. This is the "**general solution**."

Example: Let $\mathbf{A} := \begin{bmatrix} 2 & -1 & 5 \end{bmatrix}$ and $\mathbf{A}\vec{x} = 6$. Find general solution.

$\ker \mathbf{A}$ = (vectors perp. to $\begin{bmatrix} 2 & -1 & 5 \end{bmatrix}$) = (plane defined by $2x - y + 5z = 0$)

Observe $\mathbf{A}$ already reduced with c_2, c_3 free.

$$\ker \mathbf{A} = span\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

One solution to $\mathbf{A}\vec{x} = 6$ is ??

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

So, general solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}$.

Example: Let $\mathbf{A} := \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$, $\vec{b} := \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Find general solution.

Row reduce: $\left[\mathbf{A}|\vec{b}\right] \to \begin{bmatrix} 1 & -2 & 0 & | & 3 \\ 0 & 7 & 1 & | & -7 \end{bmatrix} \to \ldots \to \begin{bmatrix} 1 & 0 & \frac{2}{7} & | & 1 \\ 0 & 1 & \frac{1}{7} & | & -1 \end{bmatrix}$. (one solution?)

So, c_3 free. If $c_3 = 0$, then $c_1 = 1$ and $c_2 = -1$.

So one solution is: $\vec{x}^* = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Next, find $\ker \mathbf{A}$. General solution to $\mathbf{A}\vec{c} = \vec{0}$.

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} -\frac{2}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix}.$$

Thus, general solution to $\mathbf{A}\vec{c} = \vec{b}$ is: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{2}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix}$.

Example: Characterize the image and kernel of $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$.

Example: Characterize the image and kernel of $\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -2 & 1 & 1 \\ 1 & 2 & -3 & 1 \end{bmatrix}$.

Proposition: Given $\mathbf{A}^{m \times n}$, the following conditions are equivalent:

- $\ker \mathbf{A} = \{\vec{0}\}$, i.e., the homogeneous system $\mathbf{A}\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$.
- $rank\,\mathbf{A} = n$.
- $\mathbf{A}\vec{x} = \vec{b}$ has no free variables.
- $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in img\,\mathbf{A}$.

Proposition: Given $\mathbf{A}^{n \times n}$(square), the following conditions are equivalent:

- $\mathbf{A}$ is nonsingular.
- $rank\,\mathbf{A} = n$.
- $\ker \mathbf{A} = \{\vec{0}\}$
- $img\,\mathbf{A} = \mathbb{R}^n$.

The Superposition Principal

For homogeneous systems $\mathbf{A}\vec{x} = \vec{0}$, superposition allows one to generate new solutions by combining known solutions.

For inhomogeneous systems $\mathbf{A}\vec{x} = \vec{b}$, superposition combines the solutions corresponding to different inhomogeneities $\vec{b}_i$.

In physical applications, the inhomogeneities typically represent external forces $\vec{f}$, and solutions $\vec{x}$ represent the responses of the physical apparatus. The linear superposition principle says that if we know how the system responds to the *individual* forces ($\vec{f}, \vec{g}$, etc.), we immediately know it's response to any combination thereof.

Suppose $\vec{x}_1^*, \vec{x}_2^*$ are solutions to two inhomogeneous systems, $\mathbf{A}\vec{x} = \vec{b}_1$, and $\mathbf{A}\vec{x} = \vec{b}_2$, respectively

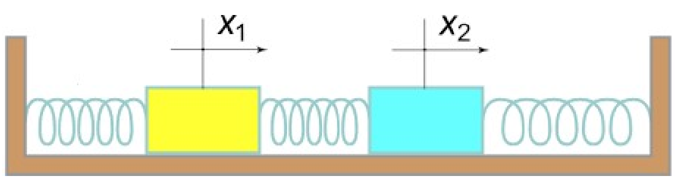
(with the same coefficient matrix $\mathbf{A}$).

Consider the system $\mathbf{A}\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$, where the RHS is a superposition of the previous two.

Then a particular solution to the combined system is given by the same superposition of the previous solutions:

$\vec{x}^* = c_1\vec{x}_1^* + c_2\vec{x}_2^*$.

Proof: $\mathbf{A}\vec{x}^* = \mathbf{A}\left(c_1\vec{x}_1^* + c_2\vec{x}_2^*\right) = c_1\mathbf{A}\vec{x}_1^* + c_2\mathbf{A}\vec{x}_2^* = c_1\vec{b}_1 + c_2\vec{b}_2$.



Spring-Mass Set-up

Example: The system: $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ models the mechanical response of a pair of masses

connected by springs, subject to external forcing $\vec{f} \in \mathbb{R}^2$ (constant acceleration).

Solution $\vec{x} = (x_1, x_2)^T$ represents displacements of masses, while entries of RHS $\vec{f} = (f_1, f_2)$ are applied forces.

We can directly determine the response of the system $\vec{x}_1^* = \left(\frac{4}{15}, -\frac{1}{15}\right)$ to a unit force $\hat{e}_1 = (1,0)$ on the first mass,

and the response $\vec{x}_2^* = \left(-\frac{1}{15}, \frac{4}{15}\right)$ to a unit force $\hat{e}_2 = (0,1)$ on the second mass.

Superposition now gives response $\vec{x}$ of system to *any* general force $\vec{f}$ since:

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_1\hat{e}_1 + f_2\hat{e}_2 = f_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + f_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and hence}$$

$$\vec{x} = f_1\vec{x}_1^* + f_2\vec{x}_2^* = f_1\begin{bmatrix} \frac{4}{15} \\ -\frac{1}{15} \end{bmatrix} + f_2\begin{bmatrix} -\frac{1}{15} \\ \frac{4}{15} \end{bmatrix} = \begin{bmatrix} \frac{4}{15}f_1 - \frac{1}{15}f_2 \\ -\frac{1}{15}f_1 + \frac{4}{15}f_2 \end{bmatrix}.$$

Theorem: Suppose that $\vec{x}_1^*, \ldots, \vec{x}_k^*$ are particular solutions to each of the inhomogeneous linear systems $\mathbf{A}\vec{x} = \vec{b}_1, \quad \mathbf{A}\vec{x} = \vec{b}_2, \quad \ldots \quad \mathbf{A}\vec{x} = \vec{b}_k$, all having the same coefficient matrix $\mathbf{A}$, and where $\vec{b}_1, \ldots, \vec{b}_k \in img\,\mathbf{A}$. Then, for any choice of scalars $c_1, \ldots, c_k$, a particular solution to the combined system $\mathbf{A}\vec{x} = c_1\vec{b}_1 + \ldots + c_k\vec{b}_k \qquad (*)$ is the corresponding superposition: $\vec{x}^* = c_1\vec{x}_1^* + \ldots + c_k\vec{x}_k^*$ of individual solutions. The general solution to $(*)$ is $\vec{x} = \vec{x}^* + \vec{z} = c_1\vec{x}_1^* + \ldots + c_k\vec{x}_k^* + \vec{z}$, where $\vec{z} \in \ker\mathbf{A}$ is the general solution to the homogeneous system $\mathbf{A}\vec{z} = \vec{0}$.

Therefore, if we know particular solutions $\vec{x}_1^*, \ldots, \vec{x}_m^*$ to $\mathbf{A}\vec{x} = \hat{e}_i$ for each $i = 1, \ldots, m$, then we can reconstruct a particular solution $\vec{x}^*$ to the general linear system $\mathbf{A}\vec{x} = \vec{b}$, by first writing: $\vec{b} := b_1\hat{e}_1 + \ldots + b_m\hat{e}_m$ as a linear combination of the basis vectors, and then using superposition to form $\vec{x}^* = b_1\vec{x}_1^* + \ldots + b_m\vec{x}_m^*$.

Example: Find a solution $\vec{x}_1^*$ to the system $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

and a solution $\vec{x}_2^*$ to $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Express the solution to $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as a linear combination of $\vec{x}_1^*$ and $\vec{x}_2^*$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 10 & 3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{10} & \frac{1}{10} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{10} & -\frac{2}{10} \\ 0 & 1 & \frac{3}{10} & \frac{1}{10} \end{array}\right].$$

So: $\vec{x}_1^* = \begin{bmatrix} \frac{4}{10} \\ \frac{3}{10} \end{bmatrix}$ and $\vec{x}_2^* = \begin{bmatrix} -\frac{2}{10} \\ \frac{1}{10} \end{bmatrix}$.

Observe that $\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, therefore the particular solution we are seeking is

$\vec{x}_3^* = 1\vec{x}_1^* + 4\vec{x}_2^* = \begin{bmatrix} \frac{4}{10} \\ \frac{3}{10} \end{bmatrix} + 4\begin{bmatrix} -\frac{2}{10} \\ \frac{1}{10} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{7}{10} \end{bmatrix}$. And to check our work, observe that:

$$\mathbf{A}\vec{x}_3^* = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} -\frac{2}{5} \\ \frac{7}{10} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}. \quad \checkmark$$

Adjoint Systems, Cokernel, and Coimage

Definition: The *adjoint* to $\mathbf{A}\vec{x} = \vec{b}$ of m equations in n unknowns is $\mathbf{A}^T\vec{y} = \vec{f}$ consisting of n equations in m unknowns $\vec{y} \in \mathbb{R}^m$ with RHS $\vec{f} \in \mathbb{R}^n$.

On the surface, there appears to be no direct connection between the solution sets of a linear system and its adjoint. However, the two are linked.

Definitions: The *coimage* of an $m \times n$ matrix $\mathbf{A}$ is the image of its transpose, $\textit{coimg}\,\mathbf{A} = \textit{img}\,\mathbf{A}^T = \left\{\mathbf{A}^T\vec{y} \mid \vec{y} \in \mathbb{R}^m\right\} \subset \mathbb{R}^n$.

The coimage coincides with the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$, called $\mathbf{A}'$s *row space*.

The *cokernel* of $\mathbf{A}$ is the kernel of its transpose, $\textit{co}\ker\mathbf{A} = \ker\mathbf{A}^T = \left\{\vec{w} \in \mathbb{R}^m \mid \mathbf{A}^T\vec{w} = \vec{0}\right\} \subset \mathbb{R}^m$, that is, the set of solutions to the homogeneous adjoint system.

The adjoint system $\mathbf{A}^T\vec{y} = \vec{f}$ has a solution **iff** $\vec{f} \in \textit{img}\,\mathbf{A}^T = \textit{coimg}\,\mathbf{A}$.

Observe that if $\mathbf{A}^T\vec{w} = \vec{0}$, then taking the transpose of both sides:

$(\mathbf{A}^T\vec{w})^T = \vec{0}^T \Rightarrow \vec{w}^T\mathbf{A} = \vec{0}^T$.

So if we let $\vec{r} := \vec{w}^T$, we have:

Definition: The cokernel of $\mathbf{A}$ can be identified with the set of all row vectors $\vec{r}$ satisfying $\vec{r}\mathbf{A} = \vec{0}^T$, and therefore the cokernel is referred to as the *left null space of* $\mathbf{A}$.

Example: Find the *general* solution to the adjoint of $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{bmatrix}$.

First: $\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ -7 & 5 & -2 \\ 9 & -3 & 6 \end{bmatrix}$. Then solve the system $\mathbf{A}\vec{y} = \vec{f}$.

$$\begin{bmatrix} 1 & 0 & 1 & f_1 \\ -3 & 1 & -2 & f_2 \\ -7 & 5 & -2 & f_3 \\ 9 & -3 & 6 & f_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & f_1 \\ 0 & 1 & 1 & 3f_1+f_2 \\ 0 & 5 & 5 & 7f_1+f_3 \\ 0 & 0 & 0 & 3f_2+f_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & f_1 \\ 0 & 1 & 1 & 3f_1+f_2 \\ 0 & 0 & 0 & -8f_1-5f_2+f_3 \\ 0 & 0 & 0 & 3f_2+f_4 \end{bmatrix}.$$

This requires $f_4 = -3f_2$ and $f_3 = 8f_1 + 5f_2$. We have one free column: y_3.

Satisfying these yields: $\vec{y} = \begin{bmatrix} f_1 - y_3 \\ 3f_1 + f_2 - y_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ 3f_1 + f_2 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

First term on the RHS represents a particular solution,

the second is the general element of the cokernel: $\ker \mathbf{A}^T = co\ker \mathbf{A}$.

The Fundamental Theorem of Linear Algebra

Observe that the rank of a matrix r (# of pivots), indicates # of independent columns, but also # of independent rows! Therefore:

Theorem: Given $\mathbf{A}^{m\times n}$, let r be its rank. Then, $\dim coimg\,\mathbf{A} = \dim img\,\mathbf{A} = rank\,\mathbf{A} = rank\,\mathbf{A}^T = r$,

$\dim \ker \mathbf{A} = n - r$, $\dim co\ker \mathbf{A} = m - r$.

Proof in book.

$\mathbf{A}$ and $\mathbf{A}^T$ have same rank, even though their row echelon forms are quite different and almost never transposes of each other.

Basis for Subspaces

Given $\mathbf{A}^{m\times n}$ with row echelon form $\mathbf{U}$, to find a basis for:

- $img\,\mathbf{A}$: choose r columns of $\mathbf{A}$ in which the pivots appear in $\mathbf{U}$ (col space);
- $\ker \mathbf{A}$: write gen. sol. to $\mathbf{A}\vec{x} = \vec{0}$ as a linear combination of $n - r$ basis vectors whose coefficients are the free vars;
- $coimg\,\mathbf{A}$: choose r nonzero rows of $\mathbf{U}$ (row space);
- $co\ker \mathbf{A}$: write gen. sol. to adjoint system $\mathbf{A}^T\vec{y} = \vec{0}$ as linear combination of $m - r$ basis vectors whose coefficients are the free vars.

Example: Find dimension of, and a basis for, the subspace spanned by the following set of vectors:

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Three pivot points imply the image is 3D, spanned by the first, second, and fourth columns of **A**.

Therefore, a basis is: $\vec{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}.$