2.4 Basis and Dimension

Definition: A basis $\mathcal{B}$ of a vector space V is a finite collection of elements $\vec{v}_1, \ldots, \vec{v}_n \in V$ that spans V, and is linearly independent.

Example: $\mathcal{B} := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ basis for $\mathbb{R}^2$. But so is ...

$\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$, as is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$, etc.

Theorem: Every basis $\mathcal{B}$ of $\mathbb{R}^n$ consists of exactly n vectors. Furthermore, a set of n vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is a basis **iff** the $n \times n$ matrix $\mathbf{A} = \begin{pmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{pmatrix}$ is nonsingular; in other words $rank\,\mathbf{A} = n$.

Theorem: Suppose vector space V has a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ for some $n \in N$. Then every other basis of V has the same number, n, of elements in it. This number is called the **dimension of** V, and is written $\dim V = n$.

The proof of this theorem rests on the following lemma:

Lemma: Suppose $\vec{v}_1, \ldots, \vec{v}_n$ span a vector space V. Then every set of $k > n$ elements $\vec{w}_1, \ldots, \vec{w}_k \in V$ is linearly dependent.

Proof of Lemma: We can write each element $\vec{w}_j = \sum_{i=1}^{n} a_{ij}\vec{v}_i$ (where $j = 1, \ldots, k$), as a linear combination of the spanning set.

Then, $c_1\vec{w}_1 + \ldots + c_k\vec{w}_k = c_1 \sum_{i=1}^{n} a_{i1}\vec{v}_i + \ldots + c_k \sum_{i=1}^{n} a_{ik}\vec{v}_i = \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij}c_j\vec{v}_i$.   (collected the $\vec{v}_i$)

It is sufficient to prove the lemma to show that $c_1\vec{w}_1 + \ldots + c_k\vec{w}_k = \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij}c_j\vec{v}_i = \vec{0}$ has a nontrivial

$\vec{c} = (c_1, \ldots, c_k)$ solution. Looking at the sigma eq term-wise, each of the n terms' coefficients will be zero when

$\sum_{j=1}^{k} a_{ij}c_j = 0$ (where $i = 1, \ldots, n$). Observe this consists of n equations in $k > n$ unknowns c_j.

A previous theorem guarantees that every homogeneous system with more unknowns than equations always has a nontrivial solution $\vec{c} \neq \vec{0}$, and this immediately implies that $\vec{w}_1, \ldots, \vec{w}_k$ are linearly dependent. ■

Proof of the preceding theorem (every basis has same number of elements):

Recall: $\vec{v}_1, \ldots, \vec{v}_m$ in $V \subseteq \mathbb{R}^n$ form a basis of V if they span V and are linearly independent.

Let $\vec{v}_1, \ldots, \vec{v}_p$ and $\vec{w}_1, \ldots, \vec{w}_q$ be bases of V.

Since $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent and $\vec{w}_1, \ldots, \vec{w}_q$ span V, we have $p \leq q$, by previous thm.

Likewise, since $\vec{w}_1, \ldots, \vec{w}_q$ are linearly independent and $\vec{v}_1, \ldots, \vec{v}_p$ span V, we have $q \leq p$. Therefore, $p = q$. ■

Examples - Find of the span, basis, dimension of the following:

- $A : \{y = x\} \subset \mathbb{R}^2$ $\ldots$

$$A = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \quad \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \quad \dim(A) = 1.$$

- B: any line in $\mathbb{R}^2$ through $\vec{0}$. $\ldots$

$B = span\{\vec{u}\}$, for some $\vec{u} \neq \vec{0}$ on the line, $\mathcal{B} = \{\vec{u}\}$, $\dim(B) = 1$.

- C: any plane in $\mathbb{R}^3$ through $\vec{0}$. $\ldots$

Spanned by two ind. vecs in plane, which also forms basis $\Rightarrow$ $\dim(C) = 2$.

- D : $span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ 42 \\ 0 \end{bmatrix}\right\} =$ $\ldots$

$$= span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \Rightarrow \dim(D) = 2.$$

- E : $\mathbb{R}^n$ $\ldots$

$$span\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}, \quad \mathcal{B} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}, \ \dim(\mathbb{R}^n) = n.$$

Suppose V is an n-dimensional vector space. Then you should be able to convince yourself that:

- Every set of more than n elements of V is linearly dependent.
- No set of fewer than n elements spans V.
- A set of n elements forms a basis **iff** it spans V
- A set of n elements forms a basis **iff** it is linearly independent.

How can you show/prove these results notationally?

Lemma: The elements $\vec{v}_1, \ldots, \vec{v}_n$ form a basis of V **iff** every $\vec{x} \in V$ can be written *uniquely* as a linear combination of the basis elements: $\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \sum_{i=1}^{n} c_i\vec{v}_i$.

Proof: The fact that basis spans V implies that every $\vec{x} \in V$ can be written as some linear combination of the basis elements.

Suppose we can write an element $\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = d_1\vec{v}_1 + \ldots + d_n\vec{v}_n$ $\quad (*)$

as two *different* combinations. Subtracting one from the other, we obtain: $(c_1 - d_1)\vec{v}_1 + \ldots + (c_n - d_n)\vec{v}_n = \vec{0}$.

The left-hand side is a linear combination of the basis elements, and hence vanishes **iff** all of its coefficients $c_i - d_i = 0$,

meaning that the two linear combinations $(*)$ are one and the same. ■