2.3 Span and Linear Independence

Definition: Let $\vec{v}_1, \ldots, \vec{v}_k$ be elements of a vector space V. A sum the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \sum_{i=1}^{k} c_i\vec{v}_i$, where the coefficients $c_1, \ldots, c_k$ are any scalars, is known as a *linear combination* of the elements $\vec{v}_1, \ldots, \vec{v}_k$. Their span is the subset $W = span\{\vec{v}_1, \ldots, \vec{v}_k\} \subset V$ consisting of all possible linear combinations with scalars $c_1, \ldots, c_k \in \mathbb{R}$.

Proposition: The span $W = span\{\vec{v}_1, \ldots, \vec{v}_k\}$ of any finite collection of vector space elements $\vec{v}_1, \ldots, \vec{v}_k \in V$ is a subspace of the underlying vector space V.

Proof: Recall $W = span\{\vec{v}_1, \ldots, \vec{v}_m\} = \left\{c_1\vec{v}_1 + \ldots + c_m\vec{v}_m \mid c_i \in \mathbb{R}\right\}$.

a) $\vec{0} \in W$?

Set $c_1 = c_2 = \ldots = c_m = 0$. Then,

$c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \vec{0}$.

b) Closed under addition?

Let $\vec{a}, \vec{b} \in W$. Then, there exist constants a_i, b_i such that:

$\vec{a} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m$,
$\vec{b} = b_1\vec{v}_1 + \ldots + b_m\vec{v}_m$.

Therefore: $\vec{a} + \vec{b} = (a_1 + b_1)\vec{v}_1 + \ldots + (a_m + b_m)\vec{v}_m \in W$.

c) Closed under scalar multiplication?

If $\vec{a} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m$ and $k \in \mathbb{R}$,

$k\vec{a} = (ka_1)\vec{v}_1 + \ldots + (ka_m)\vec{v}_m \in W$. ■

Linear Independence and Dependence

Definition: The vector space elements $\vec{v}_1, \ldots, \vec{v}_k \in V$ are called linearly dependent if there exists scalars $c_1, \ldots, c_k$, *not all zero*, such that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$. Elements that are not linearly dependent are called linearly independent.

❗ If $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent, it doesn't mean every $\vec{v}_i$ can be written as a combination of the others.

It implies at least one $\vec{v}_i$ can.

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix} \right\}$ ...

$7\vec{v}_1 + 5\vec{v}_2 + 0\vec{v}_3 = \vec{v}_4.$ However, $\vec{v}_3$ can't be written in terms of the others.

Thm: Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ and let $\mathbf{A}^{n \times k} = \left(\vec{v}_1 \ \ldots \ \vec{v}_k \right)$ be the corresponding matrix whose columns are the given vectors.

- Vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ are linearly dependent **iff** there is a nonzero solution $\vec{c} \neq \vec{0}$ to the homogeneous $\mathbf{A}\vec{c} = \vec{0}$.

- Vectors are linearly independent **iff** the only solution to the homogeneous system $\mathbf{A}\vec{c} = \vec{0}$ is the trivial one, $\vec{c} = \vec{0}$.

- Vector $\vec{b}$ lies in the span of $\vec{v}_1, \ldots, \vec{v}_k$ **iff** the linear system $\mathbf{A}\vec{c} = \vec{b}$ is compatible, i.e., has at least one solution.

We prove the first statement, leaving the other two as exercises for you.

Proof: The condition that $\vec{v}_1, \ldots, \vec{v}_k$ be linearly dependent is that there exists a nonzero vector $\vec{c} = (c_1, c_2, \ldots, c_k)^T \neq \vec{0}$ such that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$.

But recall that $\mathbf{A}\vec{c} = c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$.

Therefore, linear dependence requires the existence of a nontrivial solution to $\mathbf{A}\vec{c} = \vec{0}$. ■

Lemma: Any collection of $k > n$ vectors in $\mathbb{R}^n$ is linearly dependent.

Proposition: A set of k vectors in $\mathbb{R}^n$ is linearly independent **iff** the corresponding $n \times k$ matrix $\mathbf{A}$ has rank k.
In particular, this requires $k \leq n$.

Proposition: A collection of k vectors spans $\mathbb{R}^n$ **iff** their $n \times k$ matrix has rank n. In particular, this requires $k \geq n$.

Determining dependence of $\vec{v}_1, \ldots, \vec{v}_m$:

Dependent

- at least one is a linear combination of the others
- there exists at least one nontrivial "**linear relation**": $c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \vec{0}$.

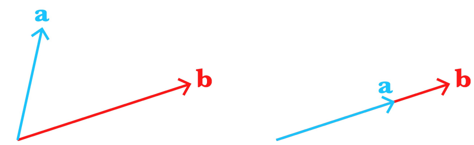
Independent

- No $\vec{v}_i$ is combination of others
- The only relation is the trivial one:

  $0\vec{v}_1 + \ldots + 0\vec{v}_m = \vec{0}$.

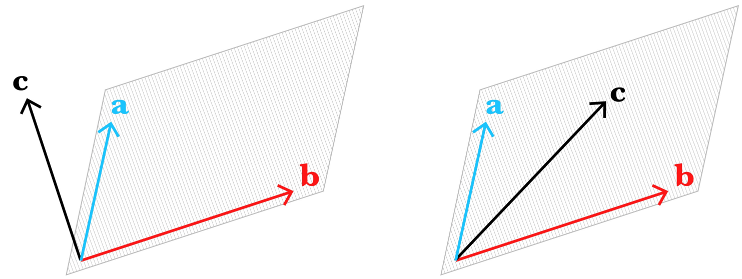
Examples:

Two vectors in $\mathbb{R}^n$ are linearly…



- Dependent if they're $\parallel$ (one is multiple of other)
- Independent if not $\parallel$.

For **three (or more) vectors**, need more than parallel criteria.



- None of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ may be parallel to the others, but vectors may still be dependent.
- Criteria is whether one is linear combination of other two.

Example: Are $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ independent?

Method 1: Look for possible relations:

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From first coordinate, c_3 must be 0. Second coordinate, c_2 must be 0. Third coordinate, $c_1 = 0$.

Only relation is trivial $\Rightarrow$ independent.

Method 2: $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 5 & 0 \end{array}\right] \overset{rref}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Method 3: Compare vectors to check for dependent system.

First vector dependent? One vector can't be dependent.

Ensure second vector not multiple of first.

Now check if third vector is a linear combination of the first two.

$$x\begin{bmatrix}0\\0\\2\end{bmatrix}+y\begin{bmatrix}0\\1\\3\end{bmatrix}=\begin{bmatrix}2\\1\\5\end{bmatrix}$$, obviously not (first component).

Example: Show that the vectors $\vec{v}_1=\begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \vec{v}_2=\begin{bmatrix}-2\\3\\-1\\1\end{bmatrix}, \vec{v}_3=\begin{bmatrix}2\\-2\\1\\-1\end{bmatrix}$ are linearly independent.

The corresponding 4×3 matrix is: $\mathbf{A} := \begin{bmatrix}1&-2&2\\0&3&-2\\2&-1&1\\1&1&-1\end{bmatrix}$.

Must show this has rank 3.

$$\to\begin{bmatrix}1&-2&2\\0&3&-2\\0&3&-3\\0&3&-3\end{bmatrix}\to\begin{bmatrix}1&-2&2\\0&3&-2\\0&0&-1\\0&0&0\end{bmatrix}\Rightarrow$$ Three pivot points imply rank 3.

Which of the following vectors are in their span? $\vec{b}_1=\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \vec{b}_2=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \vec{b}_3=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \vec{0}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

Keep in mind, the span consists of sums $c_1\begin{bmatrix}1\\0\\2\\1\end{bmatrix}+c_2\begin{bmatrix}-2\\3\\-1\\1\end{bmatrix}+c_3\begin{bmatrix}2\\-2\\1\\-1\end{bmatrix}$, for any choice of $c_i\in\mathbb{R}$.

So the zero vector $\vec{0}$ can be satisfied when the $c_i=0$.

For the others, you can attempt to solve the system: $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \vec{c} = \vec{b}.$

Observe that we can attempt to solve them all simultaneously in the following augmented matrix: $\left[\mathbf{A} \mid \vec{b}_1\ \vec{b}_2\ \vec{b}_3 \right]$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 2 & -1 & 1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 3 & -3 & 0 & -2 & 0 \\ 0 & 3 & -3 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & -3 & -2 \\ 0 & 3 & 0 & 3 & 4 & 3 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 3 & 0 & 3 & 4 & 3 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Observe that $\vec{b}_2$ has no solution due to the last row.

According to this result, $\vec{b}_1$ can be formed by: $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = 1\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3.$

And indeed: $1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$

Similarly, $\vec{b}_3$ can be formed by: $0\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3 = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Suppose (a,b,c,d) lies in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. What conditions must a,b,c,d satisfy?

Since $k_1\begin{bmatrix}1\\0\\2\\1\end{bmatrix}+k_2\begin{bmatrix}-2\\3\\-1\\1\end{bmatrix}+k_3\begin{bmatrix}2\\-2\\1\\-1\end{bmatrix}=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$, we must solve $\begin{bmatrix}1 & -2 & 2 & | & a\\0 & 3 & -2 & | & b\\2 & -1 & 1 & | & c\\1 & 1 & -1 & | & d\end{bmatrix}$

$$\rightarrow\begin{bmatrix}1 & -2 & 2 & a\\0 & 3 & -2 & b\\0 & 0 & -3 & 3c-3b-6a\\0 & 0 & 0 & -3d+3c-3a\end{bmatrix}.$$

The last row implies that $c = a + d$.

That is the only requirement on the a,b,c,d.

However, you can subsequently determine the requirements on the unique solution determined by this vector:

The 3rd row gives us $k_3 = b + a - d$.

The 2nd row gives us $3k_2 - 2k_3 = b \Rightarrow 3k_2 - 2(b + a - d) = b$

or $k_2 = b + \frac{2}{3}a - \frac{2}{3}d$.

Lastly, from the first row we have: $k_1 - 2k_2 + k_3 = a \Rightarrow k_1 - 2\left(b + \frac{2}{3}a - \frac{2}{3}d\right) + (b + a - d) = a$

$\Rightarrow k_1 = \frac{4}{3}a + b - \frac{1}{3}d$.

Therefore, every vector (a,b,c,d) such that $c = a + d$, lies in the span, and specifically:

$(a,b,c,d) = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \left(\frac{4}{3}a + b - \frac{1}{3}d\right)\vec{v}_1 + \left(b + \frac{2}{3}a - \frac{2}{3}d\right)\vec{v}_2 + (b + a - d)\vec{v}_3.$