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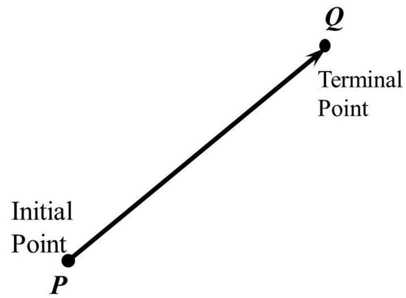
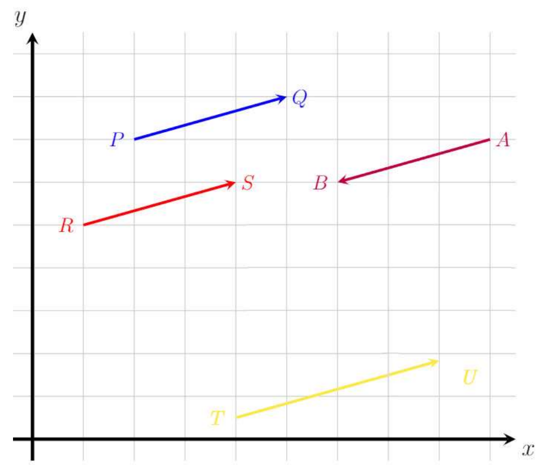
2.1 Real Vector Spaces

Vector: mathematical object consisting of **direction** & **magnitude**.



A vector, notated $\vec{v}$, $\mathbf{v}$, or $\overrightarrow{PQ}$, is often drawn as an arrow starting at **initial point** P (tail), and ending at **terminal point** Q (tip,head).

Magnitude: length of arrow.



$\vec{u}, \vec{v}$ are equivalent ($\vec{u} = \vec{v}$) if they share same magnitude & direction, even with different P (and therefore different Q).

We sometimes write $\vec{v} = \langle 2, 1 \rangle$ to mean $(2, 1)$. But $\langle 2, 1 \rangle$ refers to any vector with same magnitude/direction located anywhere in $\mathbb{R}^2$.

Zero Vector: $\mathbf{0}$ or $\vec{0}$, has length 0. Only vector with no (or any) direction.

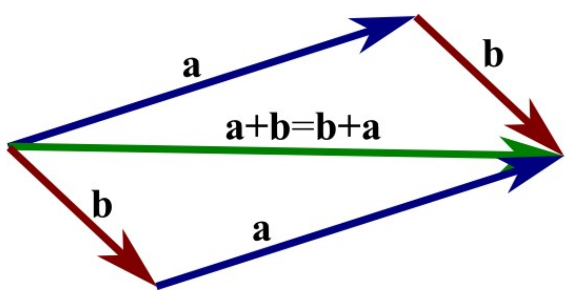
In 2D (similarly in 3D), if tail of $\vec{a}$ is at the origin, then tip has coordinates (a_1, a_2).

These are the **components of** $\vec{a}$. We write $\vec{a} = \langle a_1, a_2 \rangle$.

A *vector space* is a set V equipped with two operations:

Vector Addition (**Triangle Law**): adding any pair of vectors $\vec{a}, \vec{b} \in V$ produces another vector $\vec{a} + \vec{b} \in V$;

If $\vec{a}, \vec{b}$ are positioned with tail of $\vec{b}$ at tip of $\vec{a}$, then $\vec{a} + \vec{b}$ is from tail of $\vec{a}$ to tip of $\vec{b}$.

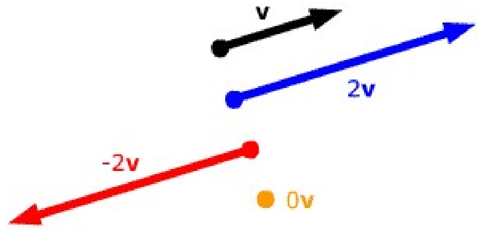


Geometric: Triangle & Parallelogram

Scalar Multiplication: multiplying a vector $\vec{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\vec{v} \in V$.

$c\vec{v} = c\langle v_1, v_2\rangle = \langle cv_1, cv_2\rangle$

Algebraic



Geometric

These are subject to the following axioms, valid for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $c, d \in \mathbb{R}$.

Commutativity of Addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Associativity of Addition: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

Additive Identity: There is a zero element $\vec{0} \in V$ satisfying $\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$.

Additive inverse: For each $\vec{v} \in V$, there is an element $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0} = (-\vec{v}) + \vec{v}$.

Distributivity: $(c + d)\vec{v} = (c\vec{v}) + (d\vec{v})$, and $c(\vec{v} + \vec{w}) = (c\vec{v}) + (c\vec{w})$.

Associativity of Scalar Multiplication: $c(d\vec{v}) = (cd)\vec{v}$.

Unit for Scalar Multiplication: the scalar $1 \in \mathbb{R}$ satisfies $1\vec{v} = \vec{v}$.

Vector spaces are not limited to literal vectors! Example: polynomials of degree 3 or less.

$P_3(x) := a_0 + a_1x + a_2x^2 + a_3x^3$, for real a_i.

A **vector space**, or more generally a **linear space** is a set of objects with a *reasonable* sense of addition and scalar multiplication.

Polynomial spaces satisfy vector space properties:

Given p_1, p_2, p_3 in P_3.

- $p_1 + p_2 = p_2 + p_1$

- $p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$

- $p_1 + 0 = 0 + p_1 = p_1$

- $p_1 + (-p_1) = (-p_1) + p_1 = 0$

- $(c+k)p_1 = cp_1 + kp_1$, for any real c, k; and $k(p_1 + p_2) = kp_1 + kp_2$, for any real k

- $c(kp_1) = (ck)p_1$

- $1(p_1) = p_1$

The following are elementary consequences of the vector space axioms:

- $0\vec{v} = \vec{0}$,

- $(-1)\vec{v} = -\vec{v}$,

- $c\vec{0} = \vec{0}$,

- If $c\vec{v} = \vec{0}$, then either $c = 0$ or $\vec{v} = \vec{0}$.

Function Spaces (Linear Spaces of Functions)

Let $F = \{$ all real valued functions on $\mathbb{R}$ $\}$; includes all polynomials, trig functions, exponentials, etc... $\ldots$

Observe for $\mathbf{f}, \mathbf{g}$ in F, we have: $(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$ and $(cf)(x) = c(f(x))$.

The properties of a linear space follow since these functions are real valued. $\ldots$

What is the neutral element? $z(x)$ $\ldots$

Linear Space of $m \times n$ Matrices

Example: 2×2 matrices. Matrices like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with neutral element $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and multiplicative identity element $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\ldots$

Infinite Dimensional Linear Spaces  

$\mathbb{R}^\infty$, space of vectors $\vec{v} = (a_1, a_2, a_3, \ldots)$, where a_i in $\mathbb{R}$.

Polynomials: $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where a_i in $\mathbb{R}$, and ALL n in $\{0, 1, 2, \ldots\}$.

Verify that they satisfy the axioms.