Announcements

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1.9 Determinants

Theoretically important in the development of linear algebra, and useful in some low dimensional, highly structured problems.

However, in practical applications, where things are high dimensional, and minimally structured, method of using determinants is rarely useful.

Nevertheless, for their theoretical importance, and completion, lets go over some basics.

Recall that the determinant of a 2×2 matrix is the following: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. More generally:

Theorem: Associated with every square matrix, there exists a unique scalar quantity, its *determinant*, that obeys the following axioms:

1. Adding a multiple of one row to another does not change the determinant.
2. Interchanging 2 rows changes the sign of the determinant.
3. Multiplying a row by any scalar (including 0) multiplies the determinant by the same scalar.
4. The determinant of any upper triangular matrix $\mathbf{U}$ is equal to the product of its diagonal entries: $\det \mathbf{U} = u_{11}u_{22}\ldots u_{nn}$.

The proof of this theorem is a result of the following results.

Lemma: Any matrix with one or more all-zero rows has determinant of zero.

Theorem: If $\mathbf{A} = \mathbf{LU}$ is a regular matrix, then: $\det \mathbf{A} = \det \mathbf{U} = u_{11}u_{22}\ldots u_{nn}$ equals the product of the pivots. More generally, if $\mathbf{A}$ is nonsingular, and requires k row interchanges to arrive at its permuted factorization $\mathbf{PA} = \mathbf{LU}$, then $\det \mathbf{A} = \det \mathbf{P} \det \mathbf{U} = (-1)^k u_{11}u_{22}\ldots u_{nn}$.

Finally, $\mathbf{A}$ is singular **iff** $\det \mathbf{A} = 0$.

Proof: In the regular case, we need only type 1 row operation to reduce $\mathbf{A}$ to upper triangular form $\mathbf{U}$, and axiom 1 says these do not change the determinant. Therefore, $|\mathbf{A}| = |\mathbf{U}|$, the formula for the latter being given by axiom 2.

The nonsingular case follows in a similar fashion. By axiom 2, each row interchange changes the sign of the determinant, and so $|\mathbf{A}|$ equal $|\mathbf{U}|$ if there has been an even number of interchanges, but equals $-|\mathbf{U}|$ if there has been an odd number.

For the same reason, the determinant of the permutation matrix $\mathbf{P}$ equal $+1$ if there has been an even number of row interchanges, and -1 for an odd number. Finally, if $\mathbf{A}$ is singular, then we can reduce it to a matrix with at least one row of zeros by type 1 and/or type 2 row operations. A previous lemma implies that the resulting matrix has 0 determinant, and so $|\mathbf{A}| = 0$, also.

Example: Compute $\det\begin{bmatrix} 5 & 6 & 6 & 6 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix}$.

$$\overset{R_1 \leftrightarrow R_2}{\Rightarrow} \begin{bmatrix} 2 & 2 & 2 & 8 \\ 5 & 6 & 6 & 6 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} \overset{\frac{1}{2}R_1}{\Rightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 5 & 6 & 6 & 6 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} \overset{\text{type 1}}{\Rightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & -4 & -16 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\overset{\text{type 1}}{\Rightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & -4 & -16 \\ 0 & 0 & 3 & 13 \end{bmatrix} \overset{-\frac{1}{4}R_3}{\Rightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 3 & 13 \end{bmatrix}$$

$$\overset{\text{type 1}}{\Rightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -14 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore: $\det\begin{bmatrix} 5 & 6 & 6 & 6 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} = (-1)^1(2 \cdot (-4)) = 8.$

Proposition: The determinant of the product of two square matrices of the same size is the product of their determinants: $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

Proposition: If $\mathbf{A}$ is a nonsingular matrix, then $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$.

Proof: Observe that the result assumes $\mathbf{A}^{-1}$ and $\mathbf{A}$ exist, and therefore we can say:

$\mathbf{AA}^{-1} = \mathbf{I}$. Then, using the above determinant property on this equation we have:

Left-Hand Side: $|\mathbf{AA}^{-1}| = |\mathbf{A}||\mathbf{A}^{-1}|$. Right-Hand Side: $|\mathbf{I}| = 1$.

So, $|\mathbf{A}||\mathbf{A}^{-1}| = 1$. And dividing by $|\mathbf{A}|$ gives us:

$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$. (recall $\mathbf{A}$ nonsingular implies $|\mathbf{A}| \neq 0$) ■

Proposition: Transposing a matrix does not change its determinant: $|\mathbf{A}^T| = |\mathbf{A}|$.

Video Tutorials (visually rich and intuitive): https://youtu.be/uQhTuRlWMxw https://youtu.be/Ip3X9LOh2dk