Announcements

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1.8 General Linear Systems

$$\begin{bmatrix} 0 & 7 & 3 & 4 & 1 \\ 0 & 0 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Zeros form a staircase

Definition: A matrix **U** is said to be in **echelon form** provided it has the following two properties (staircase properties):

- Every row of **U** that conains only zeros (if any) lies beneath every row that contains a nonzero element.
- In the other rows, the first nonzero element (**pivot**) lies strictly to the right of the first nonzero element in the preceding row (if there is a preceding row).

After reduction to echelon form, the variables corresponding to the columns with pivots, are referred to as **basic variables**.

The other variables are referred to as **free variables**.

$$\begin{bmatrix} 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So if our reduction resulted in the above echelon matrix, we would have 2 and 3 as pivots,

with x_2, x_4 as basic variables and x_1, x_3 as free variables.

Proposition: Every matrix can be reduced to row echelon form by a sequence of type 1 and/or type 2 row operations.

Definition: The *rank* of a matrix is the number of pivots. The rank of a matrix is the dimension of the vector space generated (or spanned) by its columns.

Proposition: $\mathbf{A}^{n\times n}$ is nonsingular **iff** its rank is equal to n.

Theorem: A system $\mathbf{A}\vec{x} = \vec{b}$ has either $0,\ 1,$ or ∞ solutions.

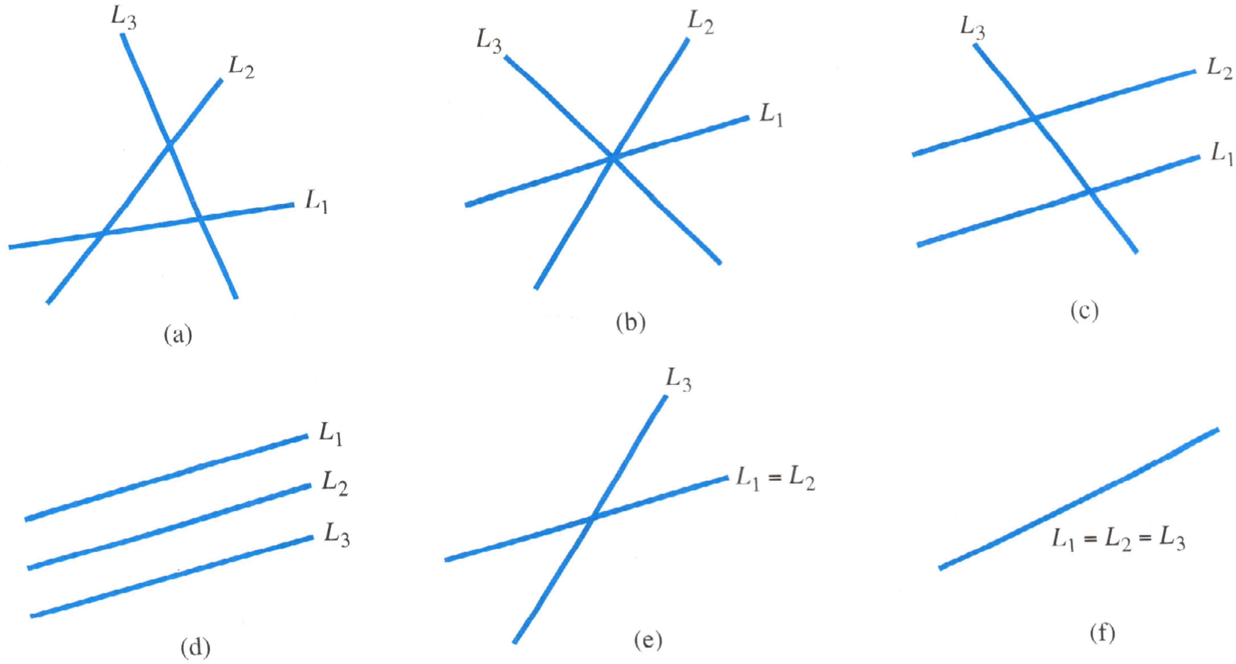
Example:

The linear system:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$

of three equations in two unknowns (x, y) represents three lines $L_1,\ L_2,$ and L_3 in the xy-plane.

The figures below show possible configurations of these lines. Describe the solution set of each:



Homogeneous Systems

Definition: A system $\mathbf{A}\vec{x} = \vec{b}$ is *homogeneous* if the $b_1, b_2, \ldots, b_n$ on the RHS are all zeros.

Otherwise, it is *nonhomogeneous*.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \end{array}\right]$$

homogeneous *nonhomogeneous*

Definition: A solution to $\mathbf{A}\vec{x} = \vec{b}$ is called **trivial** if the solution consists of $x_1 = x_2 = \ldots = x_n = 0$.

You should convince yourself that the trivial solution is always a solution to a homogeneous system.

$$\mathbf{H} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{array}\right] \Rightarrow \begin{array}{c} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \\ a_{41}x + a_{42}y + a_{43}z = 0 \end{array} \Rightarrow \begin{array}{c} a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 0 = 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 0 = 0 \\ a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 0 = 0 \\ a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 0 = 0 \end{array}$$

Observe this is true irrespective of what the a_{ij} are.

What if you knew that **H** had at least one *nontrivial* solution?
Could you conclude something about how many solutions it has?

What about a homogeneous linear system **H** with more variables (columns) than equations (rows) how many solutions?

$$\mathbf{H} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array}\right]$$

$$\rightarrow \qquad \widetilde{\mathbf{H}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & 0 \\ 0 & 1 & c_{23} & 0 \end{array} \right] \quad ??$$

Theorem: A homogeneous system $\mathbf{A}\vec{x} = \vec{0}$ of m equations in n unknowns has a nontrivial solution $\vec{x} \neq \vec{0}$ **iff** the rank of **A** is $r < n$.

If $m < n$, the system always has a nontrivial solution. If $m = n$, the system has a nontrivial solution **iff A** is singular.