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Announcements

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1.4 - 1.5 Pivoting and Permutations; Matrix Inverses

How do you solve $\mathbf{A}\vec{x} = \vec{b}$ if $\mathbf{A}$ isn't regular?

Example: $\begin{bmatrix} 0 & 1 & 2 & | & b_1 \\ 1 & 3 & 4 & | & b_2 \\ 0 & 0 & 1 & | & b_3 \end{bmatrix}$

Observe this is just the system of equations: $x_2 + 2x_3 = b_1$, $x_1 + 3x_2 + 4x_3 = b_2$, and $x_3 = b_3$.

Obviously, the order in which we listed these equations does not change the solution to the system.

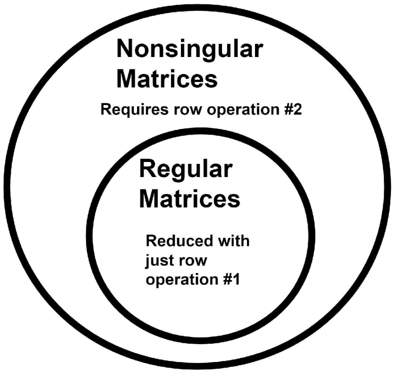
So we are allowed to list them as: $x_1 + 3x_2 + 4x_3 = b_2$, $x_2 + 2x_3 = b_1$, and $x_3 = b_3$.

This gives us the augmented matrix: $\left[\widetilde{\mathbf{A}}|\widetilde{\vec{b}}\right] = \begin{bmatrix} 1 & 3 & 4 & | & b_2 \\ 0 & 1 & 2 & | & b_1 \\ 0 & 0 & 1 & | & b_3 \end{bmatrix}$,

where $\widetilde{\mathbf{A}}$ is now regular, and this system has the same solutions as the original $\mathbf{A}$.

This justifies the row operation #2 of interchanging two rows, or pivoting.

Definition: A square matrix is called *nonsingular* if it can be reduced to upper triangular form with all nonzero diagonal elements through row operations of types #1 and #2. (i.e., add scalar multiple of one row to a lower row, and/or pivots)



Theorem: $\mathbf{A}\vec{x} = \vec{b}$ has unique solution for every choice of $\vec{b}$ **iff** $\mathbf{A}$ is square & nonsingular.

Proof of $\Leftarrow$: Nonsingularity implies reduction to upper triangular $\widetilde{\mathbf{A}}$, having same solution.
Proof of $\Rightarrow$: Section 1.8.

Observe that interchanging rows of a matrix can be accomplished by an elementary matrix, for example:

If $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, this will interchange the first two rows of a 3×3 matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix} = \begin{bmatrix} b & b & b \\ a & a & a \\ c & c & c \end{bmatrix}.$$

Definition: A permutation matrix **P** is a matrix obtained from the identity matrix by any combination of row interchanges.

Lemma: **P** is a permutation matrix **iff** each row of **P** contains all 0 entries except for a single 1, and in addition, each column of **P** also contains all 0 entries except for a single 1.

Permuted LU Factorization

Note: For nonsingular matrices, to convert them to upper triangular form, we can choose to perform the necessary pivots first, and subsequently perform the required type 1 row operations. So, then **PA** is regular, and by previous theorem can be factored as $\mathbf{PA} = \mathbf{LU}$.

How to construct the permuted LU factorization:

Start out with **A**, and two identity matrices. One will become **L**, and the other **P**.

Then, Gaussian reduce **A**, recording each pivot on the **L** matrix, and any type 1 operation on the **P** matrix.

Example: Let $\mathbf{A} = \begin{bmatrix} 0 & 2 & -5 \\ 4 & -3 & -6 \\ 2 & -2 & 0 \end{bmatrix}$. Find permutation, lower triangular, and upper triangular matrices $\mathbf{P}, \mathbf{L}, \mathbf{U}$ such that $\mathbf{PA} = \mathbf{LU}$.

So we start out with $\mathbf{L}_0 = \mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

And we notice right away that the first row of **A** will not work, so we interchange it with the 3rd row:

$$\mathbf{A}_1 = \begin{bmatrix} 2 & -2 & 0 \\ 4 & -3 & -6 \\ 0 & 2 & -5 \end{bmatrix},$$

Recording this in $\mathbf{P}$ gives us $\mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Then we proceeding with a type 1 row operation, we have:

$\mathbf{A}_2 = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -6 \\ 0 & 2 & -5 \end{bmatrix}$. Recording this in $\mathbf{L}$, gives us $\mathbf{L}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Next, another type 1 operation: $\mathbf{A}_3 = \mathbf{U} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 7 \end{bmatrix}$, recording this in $\mathbf{L}$, gives us $\mathbf{L}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Therefore: $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & -5 \\ 4 & -3 & -6 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 7 \end{bmatrix} \qquad \text{OR} \qquad \mathbf{P}_1\mathbf{A} = \mathbf{L}_2\mathbf{U}.$$

So,we have generalized LU factorization to matrices which require pivots:

Theorem: Given $\mathbf{A}^{n\times n}$. The following conditions are equivalent:

- ♦ $\mathbf{A}$ is nonsingular.
- ♦ $\mathbf{A}$ has n nonzero pivots.
- ♦ $\mathbf{A}$ admits a permuted LU factorization: $\mathbf{PA} = \mathbf{LU}$.

Matrix Inverses (§1.5)

Recall from previous mathematics that if $a \neq 0$,

then there is a (unique) number $b = a^{-1} = \frac{1}{a}$ such that $ab = ba = 1$.

We call this number its **inverse**, and we say these nonzero numbers are **invertible**.

Does this exist for matrices? Kind of, but we must change the assumption a bit.

Instead of $\mathbf{A} \neq 0$, we need something called the **determinant** of $\mathbf{A}$ to be nonzero.

Definition: We say $\mathbf{A}^{n\times n}$ is **invertible** if there exists $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

If $\mathbf{B}$ exists, then $\mathbf{B}$ is $\mathbf{A}$'s inverse, and is commonly denoted $\mathbf{A}^{-1}$.

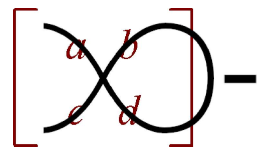
Because matrices do not commute, we must have both $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$.

In particular, **B** must be both a left- **and** a right-inverse.

But only square matrices can have both, so only square matrices can be invertible.

2×2 Matrices:

Given: $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.



A is invertible if $ad - bc$ (its **determinant**) is nonzero.

Proof: Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be it's inverse. So the right inverse condition is:

$$\mathbf{AX} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solving the system of four equations, we find: $x = \frac{d}{ad-bc}$, $y = -\frac{b}{ad-bc}$, $z = -\frac{c}{ad-bc}$, $w = \frac{a}{ad-bc}$,

provided $ad - bc \neq 0$.

In which case (in this 2×2 example) we have **inverse**: $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(One-over-Determinant, Swap, then Signs)

Theorem: $\mathbf{A}^{n \times n}$ has an inverse **iff** **A** is nonsingular. (proof provided later)

Lemma: The inverse of a square matrix, if it exists, is unique.

Proof: Suppose **X** satisfies $\mathbf{XA} = \mathbf{AX} = \mathbf{I}$ and **Y** satisfies $\mathbf{YA} = \mathbf{AY} = \mathbf{I}$.

By associativity: $\mathbf{X} = \mathbf{XI} = \mathbf{X}(\mathbf{AY}) = (\mathbf{XA})\mathbf{Y} = \mathbf{IY} = \mathbf{Y}$. ■

Lemma: If **A** is an invertible matrix, then $\mathbf{A}^{-1}$ is also invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

Proof: The matrix inverse equations $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{AA}^{-1}$ are sufficient to prove that **A** is the inverse of $\mathbf{A}^{-1}$. ■

Lemma: If **A** and **B** are invertible matrices of the same size, then their product, **AB**, is invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. Note that the order of the factors is reversed under inversion.

Proof: Let $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Then, by associativity, $\mathbf{X}(\mathbf{AB}) = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{AB} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$,

$(\mathbf{AB})\mathbf{X} = \mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$.

Thus $\mathbf{X}$ is both a left and right inverse for the product matrix $\mathbf{AB}$. ■

Warning: in general, $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

Example: Show that if $\mathbf{A}$ is a nonsingular matrix, so is every power $\mathbf{A}^n$.

Recall that nonsingular matrices are square matrices. Also, $\mathbf{A}$ is nonsingular **iff** it has an inverse $\mathbf{A}^{-1}$.

Observe that $\mathbf{A}^n(\mathbf{A}^{-1})^n = (\mathbf{AA}\ldots\mathbf{AA})(\mathbf{A}^{-1}\mathbf{A}^{-1}\ldots\mathbf{A}^{-1}\mathbf{A}^{-1}) = (\mathbf{AA}\ldots\mathbf{A})(\mathbf{AA}^{-1})(\mathbf{A}^{-1}\mathbf{A}^{-1}\ldots\mathbf{A}^{-1})$

$= (\mathbf{AA}\ldots\mathbf{A})\mathbf{I}(\mathbf{A}^{-1}\mathbf{A}^{-1}\ldots\mathbf{A}^{-1}) = (\mathbf{AA}\ldots\mathbf{A})(\mathbf{A}^{-1}\mathbf{A}^{-1}\ldots\mathbf{A}^{-1}) =\ldots= \mathbf{AA}^{-1} = \mathbf{I}$.

Therefore, $\mathbf{A}^n$ has an inverse of $(\mathbf{A}^{-1})^n$, and therefore every power of $\mathbf{A}^n$ is nonsingular.

Gauss-Jordan Elimination

To find an inverse of a nonsingular square matrix, one generally uses the *Gauss-Jordan Elimination method*.

Justification

For square matrices $\mathbf{A}$, calculating the right inverse turns out to also be the left inverse,
so we need only calculate: $\mathbf{AX} = \mathbf{I}$, where we are solving for $\mathbf{X}$.

If we write $\mathbf{X} = \begin{bmatrix} \vec{x}_1 & \ldots & \vec{x}_n \end{bmatrix}$, then recall that $\mathbf{AX} = \begin{bmatrix} \mathbf{A}\vec{x}_1 & \ldots & \mathbf{A}\vec{x}_n \end{bmatrix}$.

So, solving $\mathbf{AX} = \mathbf{I}$ amounts to solving the equations $\mathbf{A}\vec{x}_1 = \vec{e}_1, \ \ldots, \ \mathbf{A}\vec{x}_n = \vec{e}_n$ or
$\begin{bmatrix} \mathbf{A} | \vec{e}_1 \end{bmatrix}, \ \ldots, \ \begin{bmatrix} \mathbf{A} | \vec{e}_n \end{bmatrix}$, where the $\vec{e}_i$ are the standard unit vectors.

However, since each of these equations has the same coefficient matrix $\mathbf{A}$, the n calculations will
perform identical row operations.

This allows us to combine the calculations into a single calculation: $\begin{bmatrix} \mathbf{A} | \vec{e}_1 & \ldots & \vec{e}_n \end{bmatrix} = [\mathbf{A}|\mathbf{I}_n]$.

This allows us to make the same changes to the $\vec{e}_i$, but simultaneously.

So our previous method would have us reduce this to [**U**|**C**], with the task to solve this using back substitution.

However, it is usually more convenient to continue using row operations until you have obtain [**I**|**X**].

Accomplishing this may now require type 3 row operations (multiplying rows by nonzero constants).

$$\text{So, convert } [\,\mathbf{A} \mid \mathbf{I}\,] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array}\right] \text{ into…}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & a'_{11} & a'_{12} & \dots & a'_{1n} \\ 0 & 1 & 0 & 0 & a'_{21} & a'_{22} & \dots & a'_{2n} \\ 0 & 0 & 1 & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & a'_{n1} & a'_{n2} & \dots & a'_{nn} \end{array}\right] = [\,\mathbf{I} \mid \mathbf{A}^{-1}], \text{ using elementary row operations.}$$

Example: Use Gauss-Jordan elimination to find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{R2+(-3R1)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R3+(-R1)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \overset{R2\leftrightarrow R3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 6 & -5 & -3 & 1 & 0 \end{array}\right]$$

$$\overset{R3+(-6R1)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 3 & 1 & -6 \end{array}\right] =: [\mathbf{U}|\tilde{e}].$$

At this point, we have reduced the original system $\mathbf{AX} = \mathbf{I}$ to 3 equations $\mathbf{U}\vec{x} = \tilde{e}_i$
But let's continue to $[\mathbf{I}|\mathbf{A}^{-1}]$.

$$\xrightarrow{R1+2R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 3 & 1 & -6 \end{array}\right] \xrightarrow{-\frac{1}{5}R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & \frac{6}{5} \end{array}\right]$$

(notice how I am allowing myself to add constant multiples of lower rows to upper rows!)

(also notice how I avoided fractions until the last possible moment)

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & -2 \\ -5 & 0 & 5 \\ -3 & -1 & 6 \end{bmatrix}.$$

Always Be Adding

AVOID FRACTIONS!!!

Elementary Matrices: Type 3

Elementary matrix which performs scalar multiplication of i$^{\text{th}}$ row by c.

$\mathbf{E} = \begin{bmatrix} \vec{e}_1 & \dots & c\vec{e}_i & \dots & \vec{e}_n \end{bmatrix}$. With $\mathbf{E}^{-1} = \begin{bmatrix} \vec{e}_1 & \dots & \frac{1}{c}\vec{e}_i & \dots & \vec{e}_n \end{bmatrix}$

So, if we want to multiply the second column of a 3×3 matrix by 5, we can do so with: $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Verify that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is its inverse.

Lemma: Every elementary matrix is nonsingular, and its inverse is also an elementary matrix of the same type.

We now have sufficient results to prove the previously mentioned theorem:

Theorem: $\mathbf{A}^{n\times n}$ has an inverse **iff** $\mathbf{A}$ is nonsingular.

Proof: Gauss-Jordan method reduces nonsingular $\mathbf{A}^{n\times n}$ to $\mathbf{I}_n$ through row operations.

Let $\mathbf{E}_1, \dots, \mathbf{E}_N$ be the corresponding elementary matrices. So: $\mathbf{E}_N\mathbf{E}_{N-1}\dots\mathbf{E}_1\mathbf{A} = \mathbf{I}_n$.

Claim: $\mathbf{X} = \mathbf{E}_N\mathbf{E}_{N-1}\dots\mathbf{E}_1$ is the inverse of $\mathbf{A}$.

We already have that it is the left inverse, furthermore each elementary matrix has an inverse.

Therefore, $\mathbf{X}$ is itself invertible: $\mathbf{X}^{-1} = (\mathbf{E}_N\mathbf{E}_{N-1}\ldots\mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\ldots\mathbf{E}_N^{-1}$. (∗)

So, multiplying $\mathbf{XA} = \mathbf{I}$ on the left by $\mathbf{X}^{-1}$ leads to $\mathbf{A} = \mathbf{X}^{-1}$.

And by previous lemma, we also have $\mathbf{X} = \mathbf{A}^{-1}$. ■

Furthermore, if we substitute $\mathbf{A} = \mathbf{X}^{-1}$ into (∗), we get the following proposition:

Proposition: Every nonsingular matrix can be written as the product of elementary matrices.

Proposition: If $\mathbf{L}$ is a lower triangular matrix with all nonzero entries on the main diagonal, then $\mathbf{L}$ is nonsingular and its inverse $\mathbf{L}^{-1}$ is also lower triangular. In particular, if $\mathbf{L}$ is lower unitriangular, so is $\mathbf{L}^{-1}$. A similar result holds for upper triangular matrices.

Proof in textbook.

Example: Find the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$, if possible, by applying the Gauss-Jordan Method.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_2+(-2R_1)}{\Rightarrow} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_2\leftrightarrow R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\overset{R_1+R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \overset{R_1+3R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \overset{R_2+R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\overset{(-1)\cdot R_2\ \&\ (-1)\cdot R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \overset{\frac{1}{2}R_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right].$$

Therefore, $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

Solving Linear Systems with the Inverse

Theorem: If $\mathbf{A}$ is nonsingular, then $\vec{x} = \mathbf{A}^{-1}\vec{b}$ is the unique solution to the linear system $\mathbf{A}\vec{x} = \vec{b}$.

Proof: We merely multiply the system (on the left) by $\mathbf{A}^{-1}$, which yields $\vec{x} = \mathbf{A}^{-1}\mathbf{A}\vec{x} = \mathbf{A}^{-1}\vec{b}$.

Moreover, $\mathbf{A}\vec{x} = \mathbf{A}\mathbf{A}^{-1}\vec{b} = \vec{b}$, proving that $x = \mathbf{A}^{-1}\vec{b}$ is indeed the solution. ■

Example: Solve the following system of linear equations by computing the inverse of its coefficient matrix.

$3u - v = 2$ and $u + 5v = 12$.

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{A}^{-1} = \tfrac{1}{\det \mathbf{A}}\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} = \tfrac{1}{16}\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

$\Rightarrow \mathbf{A}\vec{x} = \vec{b}$, where $\vec{x} := (u, v)$ and $\vec{b} := (2, 12)$.

$$\Rightarrow \vec{x} = \mathbf{A}^{-1}\vec{b} = \tfrac{1}{16}\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{11}{8} \\ \frac{17}{8} \end{bmatrix}.$$

So, $u = \frac{11}{8}$ and $v = \frac{17}{8}$ is the unique solution. ■

The LDV Factorization

Theorem: $\mathbf{A}$ is regular **iff** it admits a factorization $\mathbf{A} = \mathbf{LDV}$, where $\mathbf{L}$ is lower unitriangular, $\mathbf{D}$ is diagonal with nonzero diagonal entries, and $\mathbf{V}$ is an upper unitriangular.

In particular, once one has calculated $\mathbf{A} = \mathbf{LU}$, then $\mathbf{D}$ is a diagonal matrix consisting of the same diagonal entries as $\mathbf{U}$, that is, the pivots. $\mathbf{V}$ is then obtained from $\mathbf{U}$ by dividing each row by its pivot.

Proposition: If $\mathbf{A} = \mathbf{LU}$ is regular, then the factors $\mathbf{L}$ and $\mathbf{U}$ are uniquely determined.

The same holds for the $\mathbf{A} = \mathbf{LDV}$ factorization.

Proof in textbook.

Theorem: $\mathbf{A}$ is nonsingular **iff** there is a permutation matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{LDV}$ (**permuted LDV factorization**), where $\mathbf{L}$ is a lower unitriangular matrix, $\mathbf{D}$ is a diagonal matrix with nonzero diagonal entries, and $\mathbf{V}$ is an upper unitriangular matrix.

Proof: Follows directly from "$\mathbf{A}$ is nonsingular **iff** $\mathbf{A} = \mathbf{LU}$" and the above proposition.

Example: Produce the **LDV** or a permuted **LDV** factorization of $\mathbf{A} = \begin{bmatrix} 2 & -2 & 0 \\ 4 & -3 & -6 \\ 0 & 2 & -5 \end{bmatrix}$.

Recall from a previous example (see above) that we had generated the **LU** factorization for **A** as:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 7 \end{bmatrix} = \mathbf{LU}.$$

Generating the diagonal **D** from **U**’s pivots: $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

Then dividing each of **U**’s rows by their pivots, we get: $\mathbf{V} = \begin{bmatrix} \frac{1}{2} \cdot \vec{r}_1 \\ 1 \cdot \vec{r}_2 \\ \frac{1}{7} \cdot \vec{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\text{Therefore: } \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{LDV}.$$