

Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ♦ Laplace transforms
- ♦ Gamma Functions
- ♦ Linearity of Transforms
- ♦ Inverse Laplace Transform
- ♦ Commonly Used Transforms
- ♦ Piecewise Continuous Functions
- ♦ Existence and Uniqueness of Laplace Transforms



10.2: Laplace Transformation of IVPs (initial value problems)

Laplace transformations can be used to solve linear non-homogeneous DEQs with constant coefficients (so linearize!).

Note that because of the linearity of Laplace transformations, when given $ax''(t) + bx'(t) + cx(t) = f(t)$, we can apply a Laplace transform to both sides of the DEQ: $a\mathcal{L}\{x''(t)\} + b\mathcal{L}\{x'(t)\} + c\mathcal{L}\{x(t)\} = \mathcal{L}\{f(t)\}$.

How does this help us solve the DEQ?

If the result of applying each Laplace transform above leaves us with an algebraic equation, we can then use our regular tools of algebra to solve the equation, and then use $\mathcal{L}^{-1}$ to solve for $x(t)$.

But how do we apply a Laplace transform to a derivative $\mathcal{L}\{x'(t)\}$?

Laplace Transforms of Derivatives Thm: if $x(t)$ is continuous and piecewise smooth for $t \geq 0$ (i.e., piecewise continuously differentiable) and is of exponential order as $t \rightarrow \infty$, then $\mathcal{L}\{x'(t)\}$ exists and $\mathcal{L}\{x'(t)\} = s\mathcal{L}\{x(t)\} - x(0)$.

$$\text{Proof}_{\text{(ish)}}: \mathcal{L}\{x'(t)\} = \int_0^{\infty} e^{-st} x'(t) dt \quad \dots$$

$$= [e^{-st} x(t)]_0^{\infty} + s \int_0^{\infty} e^{-st} x(t) dt \quad (\text{IBPs!})$$

$$= s\mathcal{L}\{x(t)\} - x(0). \quad (e^{-st} x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ for } s > c) \quad \blacksquare$$

$$\begin{aligned} \text{Similarly, } \mathcal{L}\{x''(t)\} &= s\mathcal{L}\{x'(t)\} - x'(0) = s(s\mathcal{L}\{x(t)\} - x(0)) - x'(0) \\ &= s^2\mathcal{L}\{x(t)\} - sx(0) - x'(0). \end{aligned}$$

Therefore, if we know the init-conds $x(0)$ and $x'(0)$, we can take the transform of the derivatives.

Solving a System

Given: $x'' = f(x,y,t)$ and $y'' = g(y,x,t)$, apply the Laplace transform to both DEQs as indicated above. The resulting set of equations will be an algebraic system which you can solve with our regular methods (Gaussian Reduction, Cramer's Rule, etc.).

[examples in the book]

Equation of Motion: Recall $mx''+cx'+kx = f(t)$

Here's new notation. Instead of $\mathcal{L}(x(t))$, let's write $X(s)$. (since $\mathcal{L}(x(t))$ is just some unknown function of s)

Applying the transform: $m[s^2X(s) - sx(0) - x'(0)] + c[sX(s) - x(0)] + kX(s) = F(s)$.

Or: $(ms^2 + cs + k)X(s) - mx(0)s - cx(0) - mx'(0) = F(s)$

$$\Rightarrow X(s) = \frac{mx(0)s+cx(0)+mx'(0)+F(s)}{ms^2+cs+k} = \frac{mx(0)s+cx(0)+mx'(0)}{ms^2+cs+k} + \frac{F(s)}{ms^2+cs+k} =: \frac{I(s)}{Z(s)} + \frac{F(s)}{Z(s)},$$

Note: $\frac{I(s)}{Z(s)}$ is a function of the **initial conditions** of the system,
and $\frac{F(s)}{Z(s)}$ is a function of the **external force** (non-homogeneous part).

So, this should remind you of $x(t) = x_c(t) + x_p(t)$.

Recall that $x_c(t)$ is the complementary soln of the system, while $x_p(t)$ is the particular (non-homogeneous) soln.

Therefore, apparently: $\mathcal{L}\{x(t)\} = \mathcal{L}\{x_c(t)\} + \mathcal{L}\{x_p(t)\} = \frac{I(s)}{Z(s)} + \frac{F(s)}{Z(s)}$.

Additional Laplace Transform Techniques

More transforms for our table (foun by calculating $\mathcal{L}\{x'\}$):

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}, \quad \mathcal{L}\{t \sin kt\} = \frac{2ks}{(s^2+k^2)^2}.$$

Showing the first one: Let $x(t) = te^{at}$. Then note that, $x(0) = 0$ and $x'(t) = e^{at} + ate^{at}$.

Calculating $\mathcal{L}\{x'\}$, we have $\mathcal{L}\{e^{at} + ate^{at}\} = \mathcal{L}\{x'\} = s\mathcal{L}\{x\} - 0 = s\mathcal{L}\{te^{at}\}$.

$$\Rightarrow \mathcal{L}\{e^{at}\} + a\mathcal{L}\{te^{at}\} = s\mathcal{L}\{te^{at}\} \quad (\text{linearity})$$

$$\Rightarrow \mathcal{L}\{te^{at}\} = \frac{\mathcal{L}\{e^{at}\}}{s-a} = \frac{1}{(s-a)^2}. \quad (\text{since } \mathcal{L}\{e^{at}\} = \frac{1}{s-a}) \quad \square$$

Laplace Transforms of Integral Functions

Assuming $f(t)$ is piecewise continuous for $t \geq 0$, and is of exponential order ($|f(t)| \leq Me^{ct}$), then:

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}\{f(t)\} = \frac{1}{s} F(s), \quad \text{for } s > c.$$

Equivalently, $\mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\} d\tau = \int_0^t f(\tau) d\tau$.

" $\mathcal{L}$ eats $\int_0^t d\tau$, and leaves $\frac{1}{s}$ behind. $\mathcal{L}^{-1}$ eats $\frac{1}{s}$ and leaves $\int_0^t d\tau$ behind!"

Proof: If we define $g(t) := \int_0^t f(\tau) d\tau$, then because f is piecewise continuous, the fundamental theorem of calculus implies that $g(t)$ is continuous and that $g'(t) = f(t)$ whenever f is continuous.

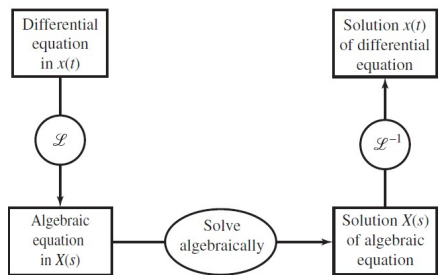
Thus, g is continuous and piecewise smooth for $t \geq 0$.

Furthermore, $|g(t)|$ is of exponential order as $t \rightarrow \infty$.

Why? Because $|g(t)| = \left|\int_0^t f(\tau) d\tau\right| \leq \int_0^t |f(\tau)| d\tau \leq \int_0^t Me^{ct} d\tau = M \frac{e^{ct}-1}{c} \leq \frac{M}{|c|} e^{ct}$ for large t .

Hence, we can apply the Derivative Thm to g ; this gives: $\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$.

Notice, $g(0) = 0$. So division by s yields: $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \mathcal{L}\{g(t)\} = \frac{1}{s} \mathcal{L}\{f(t)\} = \frac{1}{s} F(s)$. ■



Solving a DEQ w/Laplace Transforms

Example: Use Laplace transforms to solve the initial value problem:

$$x'' + 3x' + 2x = t, \quad x(0) = 0, \quad x'(0) = 2.$$

Solution: $\mathcal{L}\{x'' + 3x' + 2x\} = \mathcal{L}\{t\}$. Things I'm going to need:

$$\mathcal{L}\{x'\} = sX(s) - x(0) = sX(s).$$

$$\begin{aligned} \mathcal{L}\{x''\} &= s\mathcal{L}\{x'(t)\} - x'(0) = s(s\mathcal{L}\{x(t)\} - x(0)) - 2 \\ &= s^2X(s) - 2. \end{aligned}$$

$$\text{So, } [s^2X(s) - 2] + 3[sX(s)] + 2[X(s)] = \mathcal{L}\{t\} = ?$$

$$(s^2 + 3s + 2)X(s) - 2 = \frac{1}{s^2}$$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$

$$X(s) = \frac{2s^2+1}{s^2(s^2+3s+2)} \quad (\text{now what?})$$

In order to apply the inverse Laplace transform (since our table won't help us here), we must first use partial fractions!

$$\frac{2s^2+1}{s^2(s+1)(s+2)} = \left(\frac{A}{s} + \frac{B}{s^2}\right) + \frac{C}{s+1} + \frac{D}{s+2}.$$

Multiplying both sides by $s^2(s+1)(s+2)$:

$$2s^2 + 1 = As(s+1)(s+2) + Bs(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$$

Simplifying and collecting like terms:

$$\begin{aligned} &= (As^2 + As)(s+2) + (Bs^2 + Bs)(s+2) + (Cs^3 + 2Cs^2) + Ds^3 + Ds^2 \\ &= (As^3 + As^2) + (2As^2 + 2As) + (Bs^3 + Bs^2) + (2Bs + 2B) + (Cs^3 + 2Cs^2) + Ds^3 + Ds^2 \\ &= (A + C + D)s^3 + (3A + B + 2C + D)s^2 + (2A + 3B)s + 2B. \end{aligned}$$

Therefore, $A + C + D = 0$, $3A + B + 2C + D = 2$, $2A + 3B = 0$, $2B = 1$.

Solving four equations in four unknowns:

$$\begin{aligned} B &= \frac{1}{2}, \quad A = \frac{1}{2} \left(-\frac{3}{2}\right) = -\frac{3}{4}, \quad \left(-\frac{3}{4}\right) + C + D = 0, \quad C = \frac{3}{4} - D, \\ 3\left(-\frac{3}{4}\right) + \left(\frac{1}{2}\right) + 2\left(\frac{3}{4} - D\right) + D &= 2, \quad -\frac{9}{4} + \frac{1}{2} + \frac{6}{4} - D = 2, \quad D = -\frac{9}{4}. \\ C &= \frac{3}{4} - \left(-\frac{9}{4}\right) = 3. \end{aligned}$$

$$\text{So, } X(s) = \frac{2s^2+1}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} = \frac{\left(-\frac{3}{4}\right)}{s} + \frac{\left(\frac{1}{2}\right)}{s^2} + \frac{(3)}{s+1} + \frac{\left(-\frac{9}{4}\right)}{s+2}.$$

$$\text{And, } x(t) = \mathcal{L}^{-1}\{\mathcal{L}\{x(t)\}\} = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{-3}{s} + \frac{2}{s^2} + \frac{12}{s+1} - \frac{9}{s+2}\right)\right\} =$$

$$\frac{1}{4}\left(-3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 12\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 9\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\right). \quad (\text{linearity})$$

$$\text{Thus, } \mathcal{L}^{-1}\{X(s)\} = \frac{1}{4}(-3 + 2t + 12e^{-t} - 9e^{-2t}).$$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$

Example: Use the theorem for the Laplace transform of an integral to find the inverse Laplace transform of $F(s) = \frac{7}{s(s^2+16)}$.

Solution: Recall that:

If $f(t)$ is piecewise continuous for $t \geq 0$, and satisfies the condition of exponential order, then:

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}\{f(t)\} = \frac{1}{s} F(s) \text{ for } s > c.$$

$$\text{Equivalently, } \mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\} = \int_0^t f(\tau) d\tau.$$

Also:

$$\mathcal{L}(\sin kt) = \frac{k}{s^2+k^2} \text{ for } s > k.$$

First observe that:

$$\frac{7}{s(s^2+16)} = \frac{1}{s} \cdot \frac{7}{s^2+16} \text{ and } \mathcal{L}^{-1}\left\{\frac{7}{s(s^2+16)}\right\} = \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = \frac{7}{4}\sin(4t).$$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$

So, from the definition above...

$$\mathcal{L}^{-1}\left\{\frac{7}{s(s^2+16)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{7}{s^2+16}\right\} = \frac{7}{4} \int_0^t \sin(4\tau) d\tau$$

$$= \frac{7}{4} \left[-\frac{1}{4} \cos 4\tau \right]_{\tau=0}^t = \frac{7}{4} \left(-\frac{1}{4} \cos 4t + \frac{1}{4} \cos 0 \right) = \frac{7}{16} (1 - \cos 4t).$$

Exercises 10.2

What did we learn?

- ♦ Laplace Transform of a Derivative
- ♦ Laplace Transform of a System
- ♦ Additional Laplace Transform Techniques
- ♦ Transforms of Integral Functions



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