

Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ♦ Autonomous Nonlinear Systems
- ♦ Trajectories, critical pts, equilibrium solns for nonlinear systems
- ♦ Stability of systems



Class Notices

- ♦ None

9.2: Linear and Almost Linear Systems

Consider a 1st-order *nonlinear* non-autonomous DEQ: $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$.

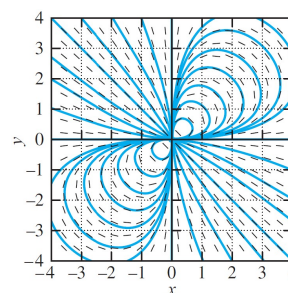
This may be impossible to solve, but by way of the chain rule: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{G(x,y)}{F(x,y)}$. (if x is differentiable and $x' \neq 0$)

Thus, the DEQ can always be recast as an autonomous *system* (studied last section): $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$.

Example: Given $\frac{dy}{dx} = \frac{2xy-y^2}{x^2-2xy}$, you can generate the autonomous system: $\frac{dy}{dt} = 2xy - y^2$, $\frac{dx}{dt} = x^2 - 2xy$.

Now that it has been transformed into a pair of polynomial DEQs, one can use an ODE plotter (like Wolfram) to plot it.

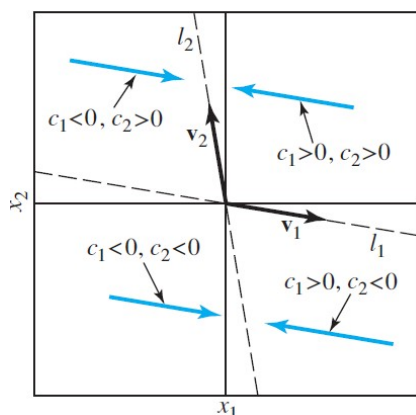
The soln curves of this system are also sols to the original non-autonomous DEQ.



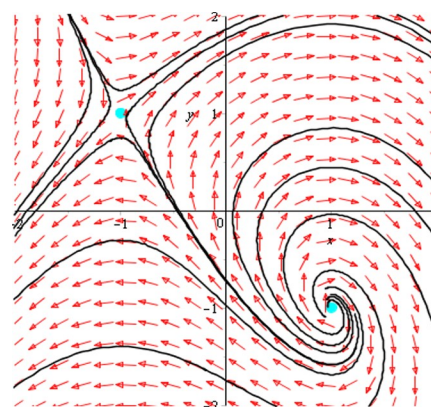
$$\frac{dy}{dx} = \frac{2xy-y^2}{x^2-2xy}$$

Now, we want to use our knowledge of the behaviors around critical pts of *linear* systems to examine the behavior of these types of *nonlinear* systems around their **isolated critical pts**.

Isolated Critical Point: A critical pt around which exists a "neighborhood" containing no other critical pts.



Previous example involving non-isolated critical pts along $\vec{v}_2$



Nonlinear DEQ with two isolated critical pts

Translating a Critical Point

Assume you have a system of the form: $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$, with an isolated critical pt located at (x_0, y_0) .

Computation is simplified if we *translate* the system so that the critical pt is now located at $(0,0)$.

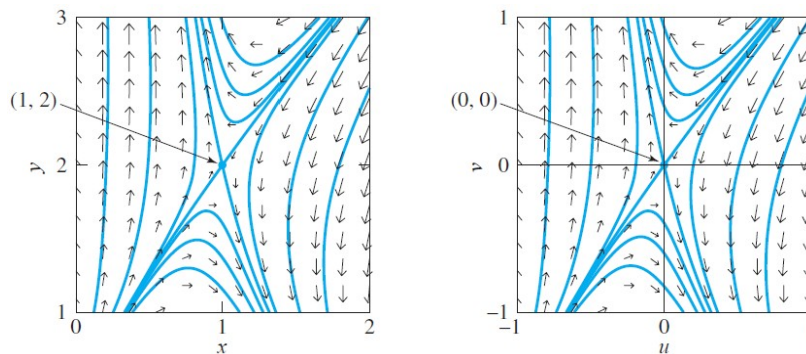
How? We make the substitution $u = x - x_0$, $v = y - y_0$.

Taking a derivative of these expressions, we have $\frac{du}{dt} = \frac{dx}{dt}$ and $\frac{dv}{dt} = \frac{dy}{dt}$. So the slopes are the same!

So our given DEQ becomes equivalent to:

$$\frac{du}{dt} = F(u + x_0, v + y_0) =: F_1(u, v).$$

$$\frac{dv}{dt} = G(u + x_0, v + y_0) =: G_1(u, v). \text{ Observe it has } (0,0) \text{ as an isolated critical pt.}$$



On left, critical pt at $(1,2)$. On right, critical pt moved to $(0,0)$.

Linearization Near a Critical Point

Note that if we let $x := u + x_0$ in $F(x,y)$ then by the chain rule:

$$F_u = \frac{\partial}{\partial u} F = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} F = \frac{\partial}{\partial u} (u + x_0) \frac{\partial}{\partial x} F = \frac{\partial}{\partial x} F = F_x.$$

Similarly, $F_v = F_y$.

Recall **Taylor's Formula** for functions of two variables $F(x,y)$ which are continuously differentiable near a fixed pt (x_0, y_0) .

$$\begin{aligned} F(x_0 + u, y_0 + v) &= F(x_0, y_0) + F_u(x_0, y_0)u + F_v(x_0, y_0)v + r(u, v) \\ &= F(x_0, y_0) + F_x(x_0, y_0)u + F_y(x_0, y_0)v + r(u, v), \end{aligned}$$

where $F(x_0, y_0)$, $F_x(x_0, y_0)$, $F_y(x_0, y_0)$ are the values of these functions at the critical pt (x_0, y_0) ,

and the "remainder term" $r(u, v)$ satisfies the condition: $\lim_{(u,v) \rightarrow (0,0)} \frac{r(u,v)}{\sqrt{u^2 + v^2}} = 0$.

Applying this formula to our two DEQs, we have:

$$\begin{aligned} \frac{du}{dt} &= F(x_0, y_0) + F_x(x_0, y_0)u + F_y(x_0, y_0)v + r(u, v), \\ \frac{dv}{dt} &= G(x_0, y_0) + G_x(x_0, y_0)u + G_y(x_0, y_0)v + s(u, v), \end{aligned} \text{ where } r(u, v) \text{ and } s(u, v) \text{ are the remainder terms.}$$

Since $r(u, v)$ and $s(u, v)$ are small near the critical pt, and $F(x_0, y_0) = G(x_0, y_0) = 0$ we can linearize the system as:

$$\begin{aligned}\frac{du}{dt} &= F_x(x_0, y_0)u + F_y(x_0, y_0)v, \\ \frac{dv}{dt} &= G_x(x_0, y_0)u + G_y(x_0, y_0)v.\end{aligned}$$

This process allows us to approximate soln curves by eliminating these nonlinear terms (**linearization**).

We call a system *almost linear* if it consists of continuously differentiable expressions, and has isolated critical points, allowing us to conduct the above transformations.

We can then rewrite this linearized system as $\vec{u}' = \mathbf{J}\vec{u}$ where $\vec{u} = (u, v)$ whose coefficient matrix is the so-called **Jacobian matrix**: $\mathbf{J}(x_0, y_0) = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix}$.

Linear Stability Review

Before using $\mathbf{J}$ to determine the stability of the nonlinear system, let's review the stability of a linear system:

$$\vec{x}' = \mathbf{A}\vec{x} \quad \text{OR} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \text{ w/e-vals } (\lambda_1, \lambda_2) \text{ of } \mathbf{A}.$$

Eigenvalues of $\mathbf{A}$	Type of Critical Point
Real, unequal, same sign	Improper node
Real, unequal, opposite sign	Saddle point
Real and equal	Proper or improper node (depending upon defect of λ)
Complex conjugate	Spiral point
Pure imaginary	Center

Stability of Linear Systems: Recall that the critical point $(0, 0)$ is:

- ♦ Asymptotically stable if the real parts of λ_1, λ_2 are both negative;
- ♦ Stable but not asymptotically stable if the real parts of λ_1, λ_2 are both zero ($\lambda_1, \lambda_2 = \pm qi$);
- ♦ Unstable if either λ_1, λ_2 has a positive real part.

Nonlinear Stability

How does linear stability relate to a *linearized system*: $\vec{u}' = \mathbf{J}\vec{u}$ OR $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ??$

Stability of Almost Linear Systems: When evaluating a nonlinear system, we can estimate a soln w/the linearized version of the system by using the e-vals λ_1, λ_2 of this linearized system.

However, since this is just an estimate, we must be careful when the stability depends on the e-vals exactly equaling each other, having exactly zero real or imaginary parts. In particular:

- ♦ If $\lambda_1 = \lambda_2$ are real numbers, then the critical pt is either a node or a spiral pt, and is asymptotically stable if $\lambda_1 = \lambda_2 < 0$, and unstable if $\lambda_1 = \lambda_2 > 0$.
- ♦ If λ_1 and λ_2 are purely imaginary, then $(0,0)$ is either a center or a spiral pt, and may be either asymptotically stable, stable, or unstable.
- ♦ Otherwise, the critical pt of the almost linear system is of the same type and stability at the critical pt of the associated linear system.

E-vals λ_1, λ_2 for Linearized System	Type of Critical Pt for Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable proper node
$\lambda_1 = \lambda_2 < 0$	Stable node <i>or spiral pt</i>
$\lambda_1 < 0 < \lambda_2$	Unstable saddle pt
$\lambda_1 = \lambda_2 > 0$	Unstable node <i>or spiral pt</i>
$\lambda_1 > \lambda_2 > 0$	Unstable proper node
$\lambda_1, \lambda_2 = a \pm bi$ ($a < 0$)	Stable spiral pt
$\lambda_1, \lambda_2 = a \pm bi$ ($a > 0$)	Unstable spiral pt
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center <i>or spiral pt</i>

Determining type and stability of a critical pt for an almost linear system: $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$.

- ♦ Find the critical pt(s) (x_0, y_0) .

- ♦ Calculate: $\mathbf{J}(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}$ for each critical pt.

This is associated w/seperate linear system(s) for each critical pt (x_0, y_0) ...

$$\frac{du}{dt} = f_x(x_0, y_0)u + f_y(x_0, y_0)v,$$

$$\frac{dv}{dt} = g_x(x_0, y_0)u + g_y(x_0, y_0)v.$$

- ♦ Calculate λ_1, λ_2 from each system using our usual methods.

- ♦ Use the above table to determine the type and stability of the associated critical pt(s).

Example: Using $\frac{dy}{dx} = \frac{3x-x^2-xy}{y+y^2-3xy}$, turn it into a non-autonomous system, then calculate it's critical pts.

Next, translate the DEQ such that $(1,2)$ is moved to the origin, then linearize the system and calculate its e-vals. Finally, determine what type of equilibrium pt the system will have near $(1,2)$.

Solution: Observe this is a rational function, so we can break it up into the non-autonomous:

$$\frac{dy}{dt} = 3x - x^2 - xy \text{ and } \frac{dx}{dt} = y + y^2 - 3xy.$$

Finding the critical pts: $0 = 3x - x^2 - xy = x(3 - x - y)$ when $x = 0$ and $x = 3 - y$.

$$0 = y + y^2 - 3xy = y(1 + y - 3x) \text{ when } y = 0 \text{ and } y = 3x - 1.$$

When $x = 0$, observe that $y = 0, -1$. Alternatively, when $y = 0$, we see that $x = 3$.

Lastly, when $x = 3 - y$ and $y = 3x - 1$, solving the last one by substitution, we have:

$$y = 3(3 - y) - 1 = 8 - 3y \Rightarrow 4y = 8 \Rightarrow y = 2 \text{ and } x = 1.$$

So we have four critical pts: $(0, 0), (0, -1), (0, 3), (1, 2)$.

Translating the DEQ: we substitute $u = x - 1$ and $v = y - 2$. That is, $x = u + 1$ and $y = v + 2$, giving:

$$3 - x - y = 3 - (u + 1) - (v + 2) = -u - v \text{ and } 1 - 3x + y = 1 - 3(u + 1) + (v + 2) = -3u + v.$$

So the system takes the form:

$$\frac{du}{dt} = F(u, v) = (u + 1)(-u - v) = -u - v - u^2 - uv,$$

$$\frac{dv}{dt} = G(u, v) = (v + 2)(-3u + v) = -6u + 2v + v^2 - 3uv, \text{ w/a critical pt at } (0, 0).$$

Linearizing: We have $\vec{u}' = \mathbf{J}\vec{u}$ where $\vec{u} = (u, v)$ when $\mathbf{J}(u_0, v_0) = \begin{bmatrix} F_u(u_0, v_0) & F_v(u_0, v_0) \\ G_u(u_0, v_0) & G_v(u_0, v_0) \end{bmatrix}$

$$= \begin{bmatrix} \frac{\partial}{\partial u}(-u - v - u^2 - uv) & \frac{\partial}{\partial v}(-u - v - u^2 - uv) \\ \frac{\partial}{\partial u}(-6u + 2v + v^2 - 3uv) & \frac{\partial}{\partial v}(-6u + 2v + v^2 - 3uv) \end{bmatrix}_{(u,v)=(0,0)}$$

$$= \begin{bmatrix} -1 - 2u - v & -1 - v \\ -6 - 3v & 2 + 2v - 3u \end{bmatrix}_{(u,v)=(0,0)} = \begin{bmatrix} -1 & -1 \\ -6 & 2 \end{bmatrix}.$$

Finding the eigenvalues: $\begin{vmatrix} -1 - \lambda & -1 \\ -6 & 2 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 8 \Rightarrow \lambda = \frac{1 \pm \sqrt{1 - 4(-8)}}{2} \Rightarrow \lambda \in (3.37, -2.37).$

Saddle pt! So, unstable.

Alternatively, we could have skipped the translation step and gone right to linearizing around $(x_0, y_0) = (1, 2)$:

We have $\vec{x}' = \mathbf{J}\vec{x}$ where $\vec{x} = (x, y)$ when $\mathbf{J}(x_0, y_0) := \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix}_{(x,y)=(x_0,y_0)}$

$$= \begin{bmatrix} \frac{\partial}{\partial x}(3x - x^2 - xy) & \frac{\partial}{\partial y}(3x - x^2 - xy) \\ \frac{\partial}{\partial x}(y + y^2 - 3xy) & \frac{\partial}{\partial y}(y + y^2 - 3xy) \end{bmatrix}_{(x,y)=(1,2)} = \begin{bmatrix} 3 - 2x - y & -x \\ -3y & 1 + 2y - 3x \end{bmatrix}_{(x,y)=(1,2)} = \begin{bmatrix} -1 & -1 \\ -6 & 2 \end{bmatrix}.$$

Exercises 9.2



What did we learn?

- ♦ Almost Linear Systems
- ♦ Isolated Critical Pt, Jacobian matrix
- ♦ Stability of Almost Linear Systems



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Materials for Other Courses Found at **MathTalker.org**