

Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ♦ Undetermined Coefficients of Systems
- ♦ Variation of Parameters of Systems



9.1: Stability and the Phase Plane

Nonlinear Autonomous Systems

Consider a nonlinear two-dimensional 1st-order system: $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$.

Since this system doesn't explicitly depend upon t (the independent variable), it's an **autonomous system**.

Our task in this section is to learn about the stability of equilibria sols for this type of *nonlinear* system.

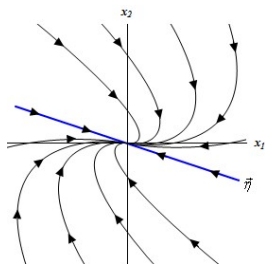
As before, soln curves $\vec{x}(t)$ to such a system in the phase plane are called a **trajectories**.

A **critical point** (x_*, y_*) is a pt that does not move w/time: $F(x_*, y_*) = 0$ and $G(x_*, y_*) = 0$.

As a result, $x(t) = x_*$ and $y(t) = y_*$ is a soln to the system, and is called an **equilibrium soln**.

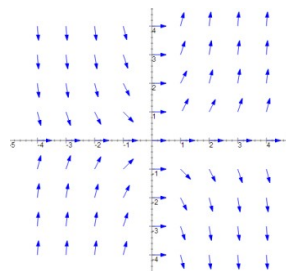
As we've seen before, the behavior of a system can be visualized using **phase portraits** or **slope fields**.

Phase Portraits



equilibria, particular sols, and flow direction

Slope/Direction Fields



slope marks

Before discussing the stability of equilibria, we must first find them!

Finding Critical Pts: Set: $F(x,y) = 0$ and $G(x,y) = 0$ and solve for x and y .
(pts where the functions don't change in time $\frac{dx}{dt} = 0$, and $\frac{dy}{dt} = 0$)

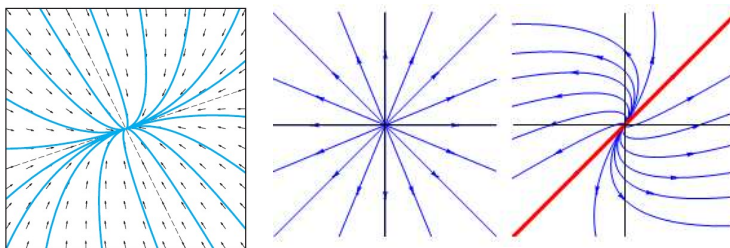
The following were defined for equilibria of *linear* DEQs in §7.4, but they also apply to *nonlinear* DEQs.

Node: Every trajectory approaches (or all recede) from (x_*, y_*) as $t \rightarrow \infty$, AND

every trajectory is tangent at (x_*, y_*) to some straight line through (x_*, y_*) . (no longer the origin)

Proper Node: Trajectories approach or recede along two independent e-vecs.

Improper Node: All trajectories approach or recede along just one e-vec.



Proper Node

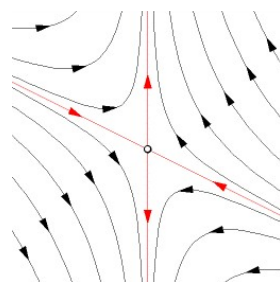
Proper Node

Improper Node

Sink: All trajectories approach the critical pt.

Source: All trajectories recede from the critical pt.

Saddle Pt: Two trajectories approach the critical pt, but all others are unbounded as $t \rightarrow \infty$.

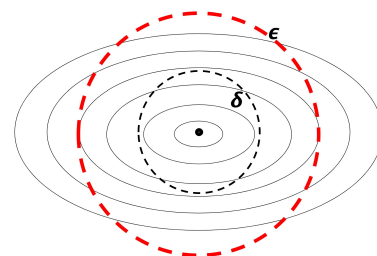


Saddle Pt

Stability of Equilibria (linear or nonlinear)

In this section we introduce the idea of stability of an equilibrium/critical pt (x_*, y_*) .

Stable Critical Pt: "If you place an init-cond $\vec{x}_0$ close enough (δ) to the critical pt (x_*, y_*) , The soln $\vec{x}(t)$ will stay close (ϵ) to (x_*, y_*) ." More formally, for each $\epsilon > 0$, there exists $\delta > 0$ such that $|\vec{x}_0 - (x_*, y_*)| < \delta$ results in $|\vec{x}(t) - (x_*, y_*)| < \epsilon$ for all $t > 0$.



stable equilibrium

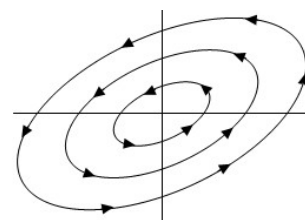
Unstable: Simply means the critical pt is not stable. (e.g., a source or saddle pt)

Closed Trajectory: Its a simple closed soln curve representing a periodic soln (like the elliptical trajectories above).

“Simple” means the trajectory (curve) is non-self-intersecting (does not cross itself).

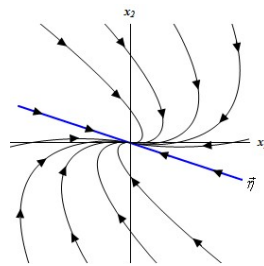
“Closed” means the soln curve in the phase plane is periodic: the system returns to the same pt after some period T , forming a loop. Equivalently: $x(t + T) = x(t)$.

Center: Is when (x_*, y_*) is stable and surrounded by simple closed trajectories.



Center

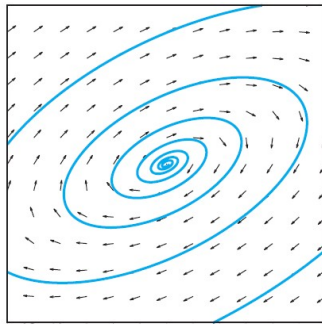
Asymptotically Stable: Every trajectory that begins sufficiently close to (x_*, y_*) , also approaches (x_*, y_*) as $t \rightarrow \infty$.



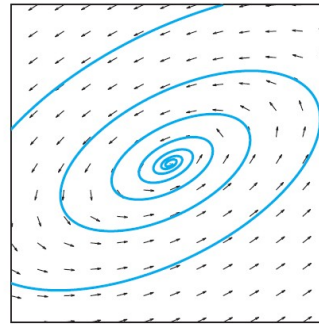
asymptotically stable

Unstable Spiral Pt (aka Spiral Source): Unstable critical pt around which trajectories spiral as they recede.

Stable Spiral Pt (aka Spiral Sink): Asymptotically stable critical pt around which trajectories spiral as they approach.



Spiral Source



Spiral Sink

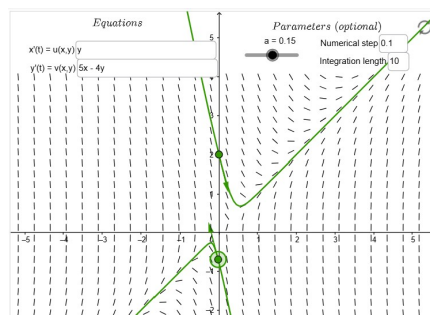
Example: Solve the following system to determine whether the critical pt $(0,0)$ is stable, asymptotically stable, or unstable. Construct a phase portrait and direction field for the given system (by hand or [Geogebra.org/m/utcMvuUy](https://www.geogebra.org/m/utcMvuUy)). Ascertain the stability or instability of this critical point, and identify it visually as a node, a saddle point, a center, or a spiral point. $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 5x - 4y$

Solution: $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}$. $\begin{vmatrix} -\lambda & 1 \\ 5 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - 5 = (\lambda + 5)(\lambda - 1) \Rightarrow \lambda \in \{1, -5\}$.

$$\lambda = 1 : \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \text{eigenvector } (x,y) = (1,1).$$

$$\lambda = -5 : \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix}, \text{eigenvector } (x,y) = (1,-5).$$

As a result, with e-vals of opposite signs, we see this is a saddle pt which is unstable.



Geogebra.org/m/utcMvuUy

Exercises 9.1

What did we learn?

- ◆ Nonlinear Autonomous Systems
- ◆ Trajectories, critical pts, equilibrium solns for nonlinear systems
- ◆ Stability of systems



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Materials for Other Courses Found at **MathTalker.org**