

Previous Lecture

- ♦ Geometry of the Sols to a DEQ
- ♦ Time Reversal
- ♦ Distinct & Repeated Real E-vals: e.g., proper & improper nodes (saddle pts)
- ♦ Complex Conjugate E-vals and E-vecs: Spiral



7.7: Numerical Methods for Systems

This is perhaps the most important section for solving DEQs occurring in real life, most of which cannot be (directly) solved with the methods presented elsewhere in this course. (!?!)

Let's say you have the IVP: $\vec{x}' = \vec{f}(t, \vec{x})$, $\vec{x}(t_0) = \vec{x}_0$, where $\vec{x} = (x_1, x_2, \dots, x_m)$ and $\vec{f} = (f_1, f_2, \dots, f_m)$.

Similar to the uniqueness and existence thm of previous class notes, as long as all of the component functions of $\vec{f}$, and their 1st-order partial derivatives are continuous in a neighborhood of $(t_0, \vec{x}_0)$, then we are guaranteed the existence and uniqueness of a soln $\vec{x} = \vec{x}(t)$ on some sub interval (of the t -axis) containing t_0 .

Once we know there is a soln, we can approximate it numerically with the following techniques.

Euler Method for Systems

Solving $\vec{x}' = \vec{f}(t, \vec{x})$, $\vec{x}(t_0) = \vec{x}_0$.

Similar to the method for a *single* DEQ, the iterative formula for a *system* is: $\vec{x}_{n+1} = \vec{x}_n + h\vec{f}(t_n, \vec{x}_n)$.

More concretely, let's look at a system of two DEQs: $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} f \\ g \end{bmatrix}$.

The IVP is: $x' = f(t, x, y)$, $x(t_0) =: x_0$,
 $y' = g(t, x, y)$, $y(t_0) =: y_0$.

So the iterative formula is:

$$x_{n+1} = x_n + hf(t_n, x_n, y_n),$$

$$y_{n+1} = y_n + hg(t_n, x_n, y_n).$$

Example: Consider the IVP:

$$x' = 3x - 2y, \quad y' = 5x - 4y$$

$$x(0) = 3, \quad y(0) = 6.$$

The exact soln is $x(t) = 2e^{-2t} + e^t$, $y(t) = 5e^{-2t} + e^t$.

Approximate the soln at $t = 0.2$ using two steps step of the Euler method.

Using the Euler method, we have: $x_{n+1} = x_n + h(3x_n - 2y_n)$, $y_{n+1} = y_n + h(5x_n - 4y_n)$.

$$\text{So, } x_1 = 3 + \frac{1}{10}(3 \cdot 3 - 2 \cdot 6) = \frac{27}{10} = 2.7,$$

$$y_1 = 6 + \frac{1}{10}(5 \cdot 3 - 4 \cdot 6) = 5.1.$$

$$x_2 = 2.7 + \frac{1}{10}(3(2.7) - 2(5.1)) = 2.49,$$

$$y_2 = 5.1 + \frac{1}{10}(5(2.7) - 4(5.1)) = 4.41.$$

The actual values at $t_2 = 0.2$ are $x(0.2) \approx 2.562$ and $y(0.2) \approx 4.573$.



There are other methods mentioned in the book (Improved Euler, Runge Kutta), which generally include more sophisticated methods of estimating the slope $\vec{f}$ necessary to accurately predict the subsequent values x_{n+1}, y_{n+1} . They do this by checking the slope of the function at several spots, instead of just at $\vec{x}_{n-1}$. Another option is varying the step size when necessary to maintain accuracy.

Exercises 7.7



What did we learn?

- ♦ Euler Method for Systems



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Materials for Other Courses Found at **MathTalker.org**