

Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

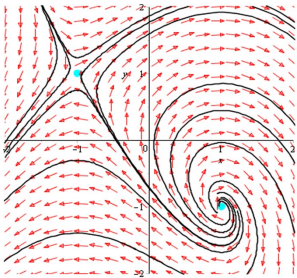
Previous Lecture

- ◆ E-val Sols of $\vec{x}' = \mathbf{A}\vec{x}$
- ◆ Steps to a Soln for $\vec{x}' = \mathbf{A}\vec{x}$.
- ◆ Steps for Complex E-vals $\lambda \in \mathbb{C}$



7.4: Solution Curves of Linear Systems

Below, you will learn how to find the behavior of soln curves $\vec{x}(t)$ for 2D *linear* systems like $\vec{x}' = \mathbf{A}\vec{x}$.



Nonlinear 2D DEQ w/2 critical pts

However, in subsequent courses, you can learn how to linearize a *nonlinear* DEQ around critical pts (the light blue dots in the image) to discover the local behavior of the soln curves. By doing this w/each of the critical pts, you can patch the behaviors together to discover the overall behavior of sols in a nonlinear system.

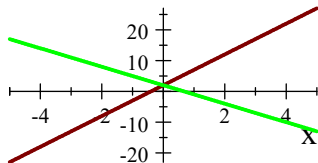
For now, we assume a linear 1st-order homogeneous constant-coefficient system:

$$\vec{x}' = \mathbf{A}\vec{x}, \text{ where } \mathbf{A} \text{ is } 2 \times 2. \text{ Let's call } \lambda_1, \lambda_2 \text{ the e-vals of } \mathbf{A}.$$

It turns out that we can predict the geometry of the sols by simply knowing λ_1, λ_2 !!

This is similar to how you previously learned the graphs of:

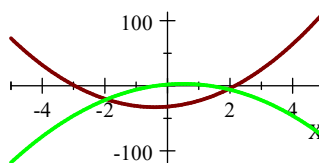
$$f_1 = cx + d,$$



Straight line. Red w/ $c > 0$.

c tells us slope

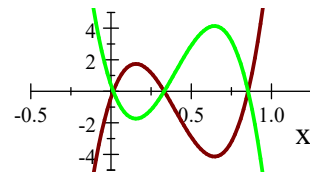
$$f_2 = cx^2 + dx + e,$$



Curve w/1 extrema. Red with $c > 0$

c tells us direction and curvature

$$\text{and } f_3 = cx^3 + dx^2 + ex + f.$$



Curve with zero or two extrema

c tells us direction and "spread"

The sign of c tells you about the shape of the graph, without needing to look at the graph.

Let's try this w/DEQs $\vec{x}' = \mathbf{A}\vec{x}$. Recall that for $\mathbf{A}$'s e-vals:

- ◆ If $\lambda, \bar{\lambda} = p \pm iq$, we get a tentative soln w/real and imaginary parts $\vec{f}(t)e^{\lambda t} + i\vec{g}(t)e^{\lambda t}$,
from which we get the real valued sols: $\vec{x}(t) = c_1\vec{f}(t)e^{\lambda t} + c_2\vec{g}(t)e^{\lambda t}$.

- ◆ If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, sols take the form: $\vec{x}(t) = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t}$.

- ◆ If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then λ_1 may not produce two e-vecs.

If $\vec{v}_1, \vec{v}_2$ are produced, then we get: $x(t) = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_1 t}$.

Otherwise, (you'd learn in §7.6) from $\vec{v}_1$, you form: $x(t) = c_1\vec{v}_1e^{\lambda_1 t} + c_2(\vec{v}_1 t + \vec{v}_g)e^{\lambda_1 t}$, where $\vec{v}_g$ is a "generalized" e-vec.

So, perhaps in analogy to the polynomials above, we can find qualitatively different sols depending upon the signs of constants λ_1, λ_2 , and also from c_1, c_2 .

E-val Scenarios

For *real* e-vals $\lambda \in \mathbb{R}$, observe we have the following possibilities:

- Distinct e-vals** $\lambda_1 \neq \lambda_2$

 - ◆ Nonzero and opposite sign ($\lambda_1 < 0 < \lambda_2$)
 - ◆ Both negative ($\lambda_1 < \lambda_2 < 0$)
 - ◆ Both positive ($0 < \lambda_2 < \lambda_1$)
 - ◆ One zero and one negative ($\lambda_1 < \lambda_2 = 0$)
 - ◆ One zero and one positive ($0 = \lambda_2 < \lambda_1$)
- Repeated e-val** $\lambda_1 = \lambda_2$

 - ◆ Positive ($\lambda_1 = \lambda_2 > 0$)
 - ◆ Negative ($0 > \lambda_1 = \lambda_2$)
 - ◆ Zero ($\lambda_1 = \lambda_2 = 0$)

Each of these situations results in different looking solution curves.

Observe that the origin ($\vec{x} \equiv \vec{0}$) is always an equilibrium soln of $\vec{x}' = \mathbf{A}\vec{x}$. However, in the cases below we explore the different soln behaviors that can occur *around* the origin.

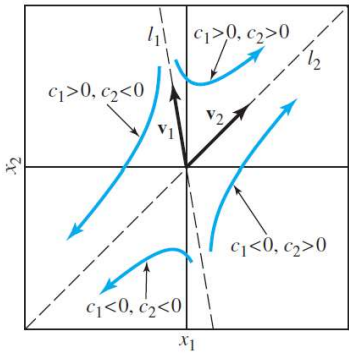
In what follows, it's helpful to remember that a positive e-val λ represents pushing sols away from the origin along the e-vec $\vec{v}$, and a negative e-val represents pulling sols toward the origin.

Saddle Points

E-vals of Opposite Sign: $\lambda_1 < 0 < \lambda_2$, $x(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$.

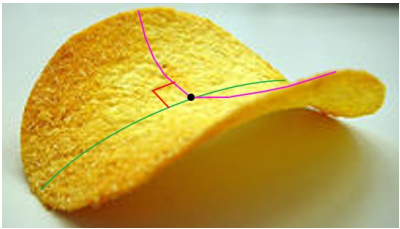
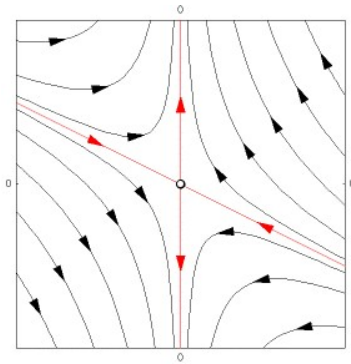
If c_1 and c_2 are nonzero, trajectories asymptote to one of two lines (l_1 or l_2) passing through the origin (see image), which are parallel to $\vec{v}_1, \vec{v}_2$.

Sols beginning on one of these lines move toward the origin as $t \rightarrow \infty$, and sols on the other line move away from the origin.



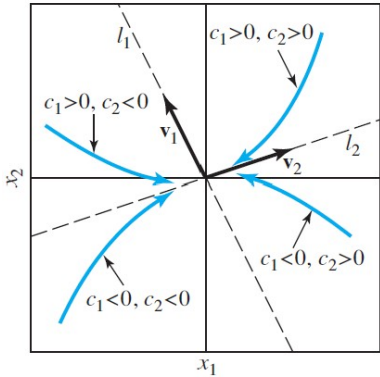
The origin itself is an equilibrium soln (critical pt), and is referred to as a **saddle point**.

Saddle Point: Two trajectories approach the critical pt, but all others are unbounded as $t \rightarrow \infty$.



Distinct Negative E-vals: $\lambda_1 < \lambda_2 < 0$, $x(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$

Every trajectory approaches the origin as $t \rightarrow \infty$.



Now that we have some examples as a frame of reference, let’s learn some terms.

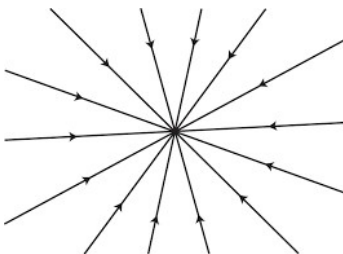
Nodes

The preceding is an example of a *node*.

Node: Every trajectory approaches (or recedes) from the origin as $t \rightarrow \infty$, *AND* every trajectory is tangent, at the origin, to some straight line through the origin.
 (Note that the saddle pt wasn’t a node.)

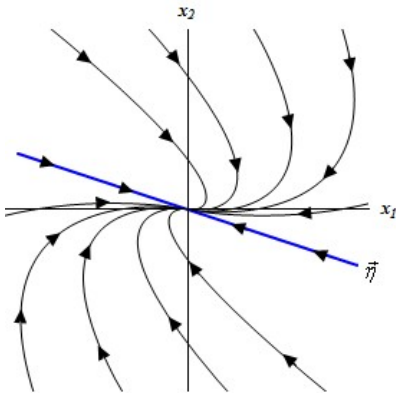
Proper Node: A node for which there are two distinct e-vecs.

A **star pt** (an example of a **proper node**) is another example of a node, where each trajectory is a straight line thru the origin. And, approaching the origin, each trajectory is tangent (at the origin) to a straight line (namely, itself; since it *is* a straight line!).



Proper Node: Trajectories approach or recede in all directions.

Improper Node: All trajectories become tangent to just one vector as they approach the origin. This occurs when there is a repeated e-val, but only one e-vec.



Improper Nodal Sink, only $\vec{v}_1$

Recall:

Sink: All trajectories approach the critical pt (the origin) as $t \rightarrow \infty$.

Source: All trajectories flow away from the critical pt (the origin) as $t \rightarrow \infty$.

Time Reversal

If $\vec{x}(t)$ is a soln of the 2D system $\vec{x}'(t) = \frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t)$, then the function $\tilde{x}(s) := \vec{x}(-t)$ is a soln of the system $\tilde{x}'(s) = \frac{d}{ds}\tilde{x}(s) = -\mathbf{A}\tilde{x}(s)$.



Proof: Note that $\tilde{x}'(s) = \frac{d}{ds}\vec{x}(s) = \frac{d}{dt} \frac{dt}{ds} \vec{x}(s) = \frac{d}{dt} \frac{d}{ds}(-s) \vec{x}(s) = \frac{d}{dt}(-1) \vec{x}(s) = -\vec{x}'(-t)$.

Therefore, the LHS of $\tilde{x}'(s) = -\mathbf{A}\tilde{x}(s)$ becomes $-\vec{x}'(-t)$, and the RHS becomes

$-\mathbf{A}\vec{x}(-t)$, giving us: $\vec{x}'(-t) = \mathbf{A}\vec{x}(-t)$, or equivalently $\vec{x}'(s) = \mathbf{A}\vec{x}(s)$.

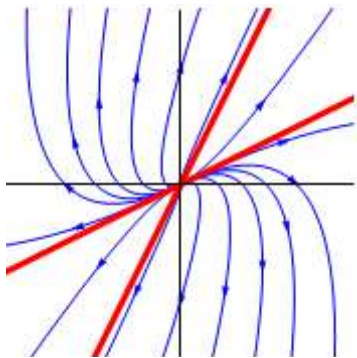
And since we know $\vec{x}'(t) = \mathbf{A}\vec{x}(t)$ for all t (even s), the equality is satisfied and $\tilde{x}(s)$ is a soln of the system $\tilde{x}' = -\mathbf{A}\tilde{x}$. ■

Relatedly, if a matrix $\mathbf{A}$ has e-vals λ_1, λ_2 , then $-\mathbf{A}$ has e-vals $-\lambda_1, -\lambda_2$

Proof: If $\mathbf{A}\vec{v} = \lambda_1 \vec{v}$, then multiplying by -1 gives us: $-\mathbf{A}\vec{v} = -\lambda_1 \vec{v}$. And similarly for λ_2 . ■

So, switching the signs of our e-vals reverses the flow of our sols.

Continuing on w/our cases: **Distinct Positive E-vals.** $0 < \lambda_2 < \lambda_1$, $x(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$ (proper node)



Proper Nodal Source

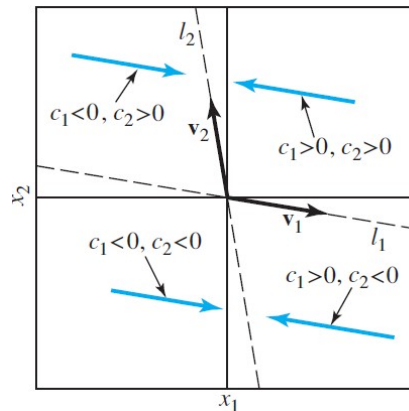
Time reversal gives us the same sols as the proper nodal source, but moving backwards in time (now a sink).

Zero E-vals and Straight-Line Sols

One Negative and One Zero E-val: $\lambda_1 < \lambda_2 = 0$, $\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2$. (moves along one vector, but not the other)

All sols are moving parallel to $\vec{v}_1$. (not a node!)

Observe below that each pt lying on the line l_2 (where $c_1 = 0$) represents a constant soln (equilibrium soln) to the system (instead of the origin being the only constant soln).



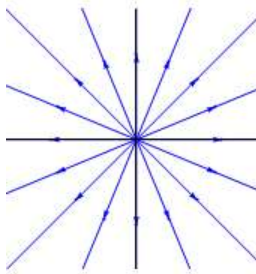
One Positive and One Zero E-val: $0 = \lambda_2 < \lambda_1$, $\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2$

By the principle of time reversal, the trajectories in this system ($\vec{x} = \mathbf{A}\vec{x}$) are identical to those in $\vec{x} = -\mathbf{A}\vec{x}$ (the image above), except that they flow in the opposite direction, away from $l_2 = \text{span}(\vec{v}_2)$, and parallel to $\vec{v}_1$.

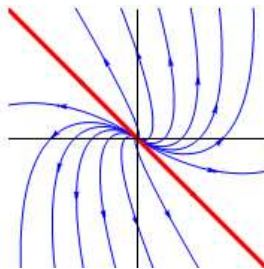
Repeated E-vals

Repeated Positive E-val: $\lambda > 0$. We have two cases:

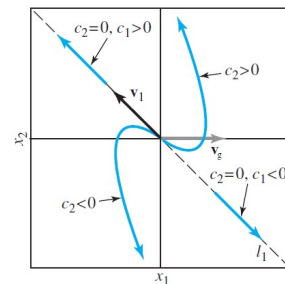
- ♦ Two Independent E-vecs: $x(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 \vec{v}_2 e^{\lambda t}$ (star or proper nodal source)
- ♦ One Independent E-vec: $x(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_g) e^{\lambda t}$ (improper nodal source)



star



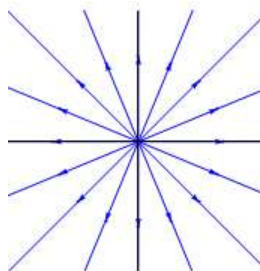
improper node



improper node

Case: Repeated Positive E-val, Two Independent E-vecs $x(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 \vec{v}_2 e^{\lambda t} = (c_1 \vec{v}_1 + c_2 \vec{v}_2) e^{\lambda t}$

We'll learn that sols for this case look like those in the graph below. (nodal star)



First, we must learn some lemmas.

Lemma: If $\mathbf{A}$ has a repeated e-val λ w/two linearly independent e-vecs, then *every* nonzero vector is an e-vec of $\mathbf{A}$ w/e-val λ .

Proof: Let $\vec{v}_1, \vec{v}_2$, be linearly independent e-vecs associated with λ . Therefore, we have $\mathbf{A}\vec{v}_i = \lambda\vec{v}_i$.

Now choose *any* $\vec{v}_0$. We must show $\mathbf{A}\vec{v}_0 = \lambda\vec{v}_0$.

Observe that since $\vec{v}_1, \vec{v}_2$ are linearly independent in $\mathbb{R}^2$, we have $\vec{v}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$, for some $c_1, c_2 \in \mathbb{R}$.

Therefore, $\mathbf{A}\vec{v}_0 = \mathbf{A}(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\mathbf{A}\vec{v}_1 + c_2\mathbf{A}\vec{v}_2$

$$= c_1\lambda\vec{v}_1 + c_2\lambda\vec{v}_2 = \lambda(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda\vec{v}_0.$$

So we have $\mathbf{A}\vec{v}_0 = \lambda\vec{v}_0$ and $\vec{v}_0$ is an e-vec of $\mathbf{A}$. ■

Lemma: If $\mathbf{A}$ has a repeated e-val λ w/two linearly independent e-vecs, then $\mathbf{A} = \lambda\mathbf{I}$.

Proof: Without loss of generality, let's choose the independent e-vecs to be $\vec{v}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

By the previous Lemma we have $\mathbf{A}\vec{v} = \lambda\vec{v}$ for every $\vec{v}$. Using $\vec{v}_1$, we have $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$,

and expanding the LHS we have $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$.

Similarly, using $\vec{v}_2$, we calculate $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$.

Therefore, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \lambda\mathbf{I}$. ■

Note that w/the previous lemma, the system $\vec{x}' = \mathbf{A}\vec{x}$ becomes:

$$\begin{aligned} x_1' &= \lambda x_1 \\ x_2' &= \lambda x_2. \end{aligned}$$

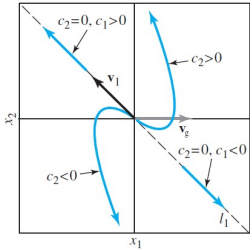
Solving these separable DEQs individually, we get:

$$\begin{aligned} x_1 &= c_1 e^{\lambda t} \\ x_2 &= c_2 e^{\lambda t} \end{aligned}$$

Or $\vec{x}(t) = e^{\lambda t} \vec{c}$, where $\vec{c}$ is solved for from init-conds. In other words, for positive e-val, the soln just pushes the init-cond vector out from the origin along a line.

Case: Repeated Positive E-val, Only One E-vec (improper node)

$$x(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_g) e^{\lambda t}, \text{ where } \vec{v}_g \text{ is a "generalized" e-vec.}$$



For the improper node, if $c_2 \neq 0$ (i.e., the init-cond is not on $\vec{v}_1$) then as $t \rightarrow -\infty$, one can discern from the vector expression (only $c_2 \vec{v}_1 t e^{\lambda t}$ survives) or the graph that each trajectory approaches the origin along a soln curve which is tangent there to $\vec{v}_1$.

Alternatively, as $t \rightarrow +\infty$, the trajectory moves far from the origin and all trajectories become parallel to $\vec{v}_1$ ($c_2 \vec{v}_1 t e^{\lambda t}$ dominates), but pointing in the opposite direction to when the soln was near the origin (where $c_2 \vec{v}_1 t e^{\lambda t}$ again dominates).

Repeated Negative E-val: $\lambda < 0$

By the time reversal property, we conclude the same properties as the positive case, except that the flow arrows reverse. Therefore we find a proper nodal (star) or improper nodal sink.

Repeated Zero E-val: $\lambda = 0$

We have 2 cases: ♦ Two (independent) E-vecs, ♦ One E-vec.

Case - Repeated $\lambda = 0$ w/Two E-vecs

Lemma: If **A** has a repeated zero e-val λ , and two e-vecs $\vec{v}_1, \vec{v}_2$, then $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Proof: Without loss of generality, let's choose the independent e-vecs to be $\vec{v}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Now observe $\mathbf{A} \vec{v}_1 = \lambda \vec{v}_1$. The left-hand side becomes $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix},$

and the right hand side becomes $0 \cdot \vec{v} = \vec{0}$.

Therefore, $a_{12} = a_{22} = 0$. A similar calculation for $\vec{v}_2$ reveals that $a_{11} = a_{21} = 0$.

The conclusion follows. ■

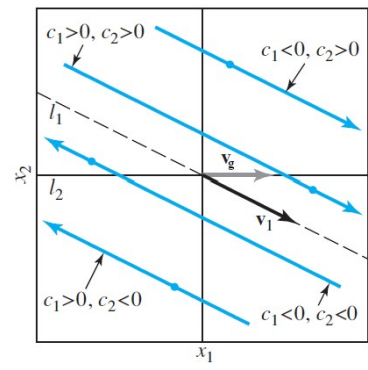
Observe that we can choose *any* vector $\vec{v}$, and $\mathbf{A} \vec{v} = \vec{0} = 0 \vec{v}$, since **A** is the zero matrix, therefore every vector is an e-vec of **A**.

Therefore, the system $\vec{x}' = \mathbf{A}\vec{x}$ becomes: $x_1' = 0$ and $x_2' = 0$, and integrating we find that the sols are constants $x_1(t) = c_1$ and $x_2(t) = c_2$, where (c_1, c_2) are simply the initial values.

In other words, each pt (c_1, c_2) in the plane becomes an equilibrium soln where the "trajectories" are the fixed pts $\vec{x}(t) \equiv (c_1, c_2)$.

Case - Repeated $\lambda = 0$ w/One E-vec: $x(t) = c_1\vec{v}_1 + c_2(\vec{v}_1t + \vec{v}_g)$, where $\vec{v}_1$ is the e-vec, and $\vec{v}_g$ is a "generalized e-vec," (explained later in the book).

In the phase plane below, note that $l_1 = \text{span}(\vec{v}_1)$ is thought of as a "median strip" dividing two opposite "lanes of traffic." (not a node!)



Complex E-vals and E-vecs

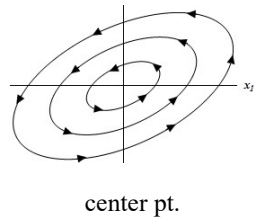
For complex e-vals, there are two basic situations:

- **Pure Imaginary E-vals:** $\lambda_1, \lambda_2 = \pm iq$.
- **Complex E-vals w/Nonzero Real Parts:** $\lambda_1, \lambda_2 = p \pm iq$, with $p \neq 0$.

Leading (from §7.3) to sols looking like: $\vec{x}(t) = c_1e^{pt}\vec{f}(t) + c_2e^{pt}\vec{g}(t)$

Pure imaginary e-vals $\lambda_1, \lambda_2 = \pm iq$ produce closed loops (not nodes!).

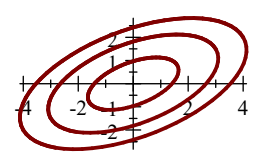
Because of the concentric nature of the sols around the origin, the origin is referred to as a **center pt.**



⚠ Knowledge of the e-vals and e-vecs of $\mathbf{A}$ are *not sufficient* to predict the direction of flow of the trajectories!

Example: take $\vec{x}' = \mathbf{A}\vec{x}$ with $\mathbf{A} = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix}$. Using our usual methods (§7.3),

we calculate sols that move *counterclockwise* as t increases.



However, if we examine $\vec{x}' = (-\mathbf{A})\vec{x}$, it has the same e-vals ($\pm 10i$) and e-vecs. And therefore the same soln curves as above, but w/sols moving *clockwise* as t increases (principle of time reversal).

So how can we know the direction of flow in advance of calculating a soln?

Lemma: For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ w/pure imaginary e-vals, if $c > 0$ the flow will be counterclockwise.
 If $c < 0$ the flow will be clockwise.

Proof: We will monitor the direction in which the soln curve flows as it crosses the positive x -axis.

If s is any positive number, then any trajectory for the system $\vec{x}' = \mathbf{A}\vec{x}$ passing through $(s, 0)$ (which sits on the positive x -axis) satisfies:

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ 0 \end{bmatrix} = \begin{bmatrix} as \\ cs \end{bmatrix} = s \begin{bmatrix} a \\ c \end{bmatrix}, \text{ at the pt } (s, 0).$$

Therefore, at $(s, 0)$ the direction of flow of the soln curve is a positive scalar multiple of the vector (a, c) .

Since c cannot be zero when $\lambda_{1,2}$ is imaginary (see problem 39 in the book), this vector either points "upward" into the 1st quadrant (if $c > 0$), or "downward" into the 4th quadrant (if $c < 0$).

So, if $c > 0$ the flow will be counterclockwise. If $c < 0$ the flow will be clockwise. ■

Complex E-vals: Spiral

Complex E-vals w/Nonzero Real Parts: $\lambda_1, \lambda_2 = p \pm iq$, with $p \neq 0$.

As w/prior cases, positive real part is associated with a *source*, and negative real part is associated with a *sink*.

General soln: $x(t) = c_1 e^{pt} (\vec{a} \cos qt - \vec{b} \sin qt) + c_2 e^{pt} (\vec{b} \cos qt + \vec{a} \sin qt)$, where $\vec{a}$ and $\vec{b}$ are the real and imaginary parts of a complex valued e-vec $\vec{v}$ of $\mathbf{A}$.

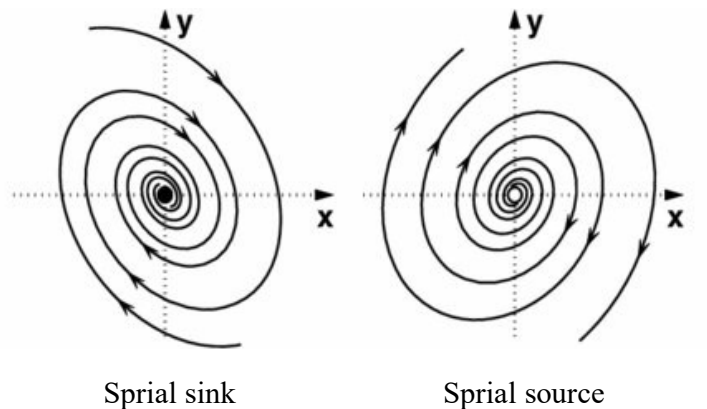
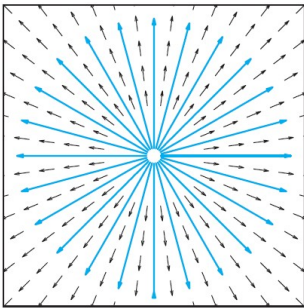
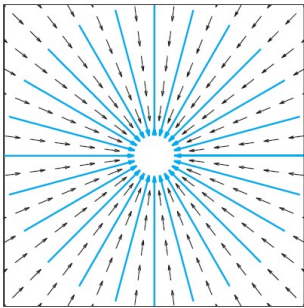


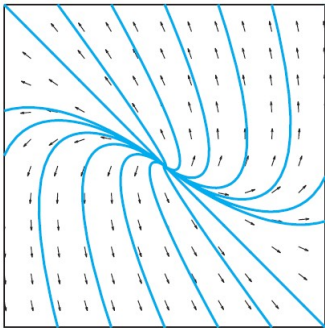
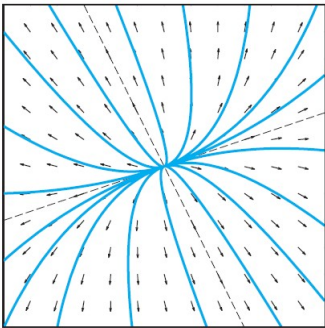
Figure 7.4.16 from the textbook



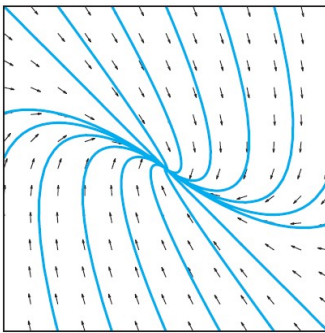
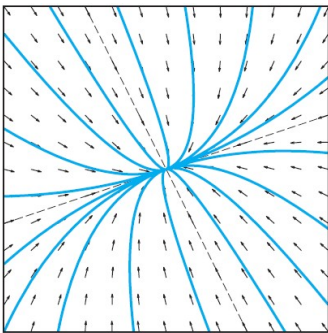
Proper Nodal Source: A repeated positive real eigenvalue with two linearly independent eigenvectors.



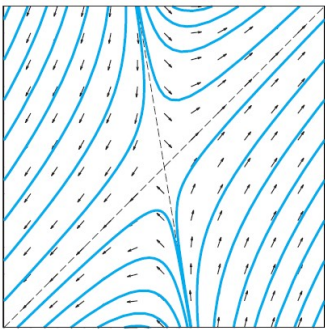
Proper Nodal Sink: A repeated negative real eigenvalue with two linearly independent eigenvectors.



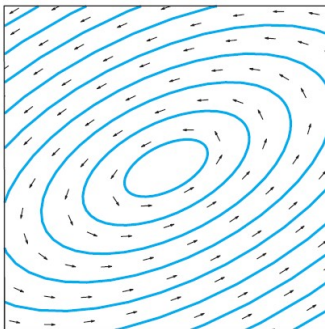
Improper Nodal Source: Distinct positive real eigenvalues (left) or a repeated positive real eigenvalue without two linearly independent eigenvectors (right).



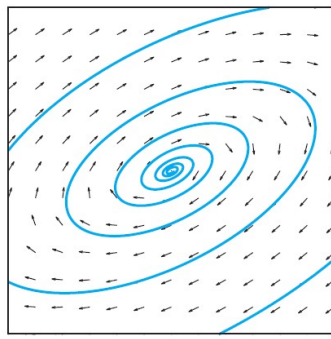
Improper Nodal Sink: Distinct negative real eigenvalues (left) or a repeated negative real eigenvalue without two linearly independent eigenvectors (right).



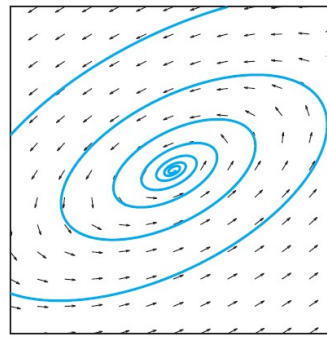
Saddle Point: Real eigenvalues of opposite sign.



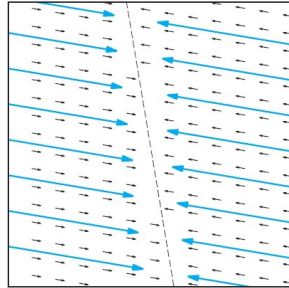
Center: Pure imaginary eigenvalues.



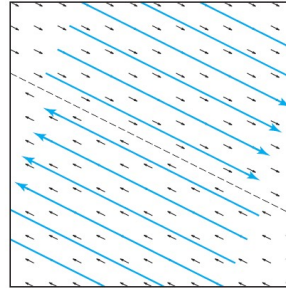
Spiral Source: Complex conjugate eigenvalues with positive real part.



Spiral Sink: Complex conjugate eigenvalues with negative real part.



Parallel Lines: One zero and one negative real eigenvalue. (If the nonzero eigenvalue is positive, then the trajectories flow away from the dotted line.)

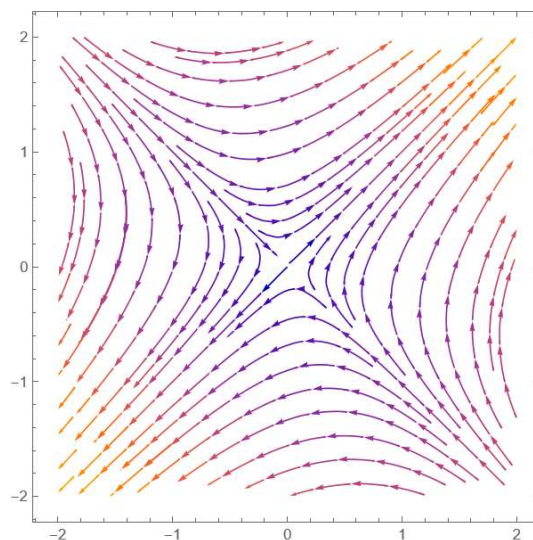


Parallel Lines: A repeated zero eigenvalue without two linearly independent eigenvectors.

Example: For the system: $x'_1 = x_1 + 3x_2$, $x'_2 = 3x_1 + x_2$, categorize the e-vals ($\lambda_{1,2} = -2, 4$) and e-vecs ($\vec{v}_1 = [-1 \ 1]^T$ and $\vec{v}_2 = [1 \ 1]^T$) of the coefficient matrix $\mathbf{A}$ according to Fig. 7.4.16 in the textbook (or above), and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator (WolframAlpha.com: `StreamPlot[{x1 + 3x2, 3x1 + x2}, {x1, -2, 2}, {x2, -2, 2}]`) to check your answer.

Solution: With real e-vals on either side of zero, we expect a saddle pt w/trajectories moving outward along the e-vec $[1 \ 1]^T$ w/the positive $\lambda = 4$, and trajectories moving inward along the e-vec $[-1 \ 1]^T$ w/the negative $\lambda = -2$.

And indeed, upon graphing we find:



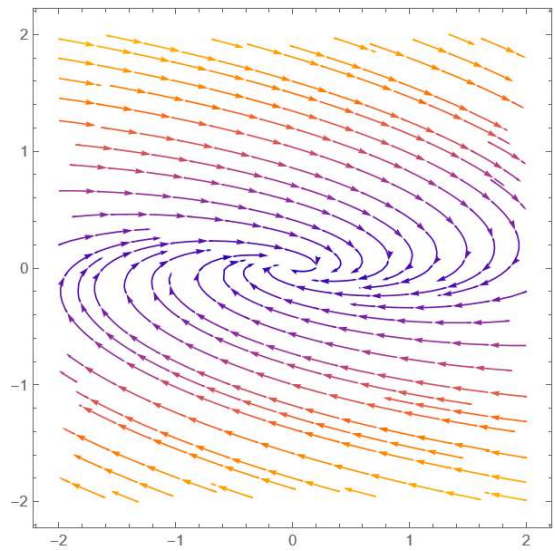
Example: For the system: $x_1' = -x_1 + 10x_2$, $x_2' = -x_1 - 3x_2$, categorize e-vals ($\lambda_{1,2} = -2 \pm 3i$) and e-vecs ($\vec{v}_{1,2} = [-1 \pm 3i \ 1]^T$) of **A** according to Fig. 7.4.16 in the textbook, and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator (WolframAlpha.com: StreamPlot[{-x1 +10x2, -x1 -3x2}, {x1, -2, 2}, {x2, -2, 2}]) to check your answer.

Solution: Observe that the real part of λ_i is less than zero, so we expect a sink.

Also, since the imaginary parts of λ_i are not zero, so we expect a spiral.

And from $\mathbf{A} = \begin{bmatrix} -1 & 10 \\ -1 & -3 \end{bmatrix}$, we see $c = -1 < 0$, so we expect the spiral to be clockwise.

And indeed, upon graphing we find:



Exercises 7.4 💪

What did we learn?

- ◆ Geometry of the Sols to a DEQ
- ◆ Time Reversal
- ◆ Distinct & Repeated Real E-vals: e.g., proper & improper nodes (saddle pts)
- ◆ Complex Conjugate E-vals and E-vecs: Spiral

