

Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ◆ Diagonals of Matrices: "Zeros off of the diagonal"
- ◆ Diagonalizing Criteria: similar matrices $\mathbf{B} = \mathbf{PAP}^{-1}$
- ◆ Diagonalizing Algorithm



6.3 - Applications Involving Powers of Matrices $\mathbf{A}^n$

Recall: if $\mathbf{A} = \mathbf{PDP}^{-1}$, then $\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\mathbf{P}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$.

Continuing in this fashion, we find: $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$ with $\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$.

Transition Matrices

A matrix $\mathbf{A}$ which "transitions" the state of a system (notated as a vector $\vec{x}_k$) from one moment $\vec{x}_k$, to the next $\vec{x}_{k+1}$.

So, $\vec{x}_{k+1} = \mathbf{A}\vec{x}_k$ where $\vec{x}_0$ is the **initial vector**.

Therefore, $\vec{x}_k = \mathbf{A}^k\vec{x}_0$. Because, $\vec{x}_k = \mathbf{A}\vec{x}_{k-1} = \mathbf{A}^2\vec{x}_{k-2} = \dots = \mathbf{A}^k\vec{x}_0$.

Predator-Prey Models

Animal "interactions" can be modeled in this way, for instance, if we let:

F := Foxes, R := Rabbits, then the populations change as

$$\vec{x} = \begin{bmatrix} F \\ R \end{bmatrix} = \begin{bmatrix} 1,500 \text{ } \img alt="fox icon" data-bbox="815 515 855 555" \\ 10,000 \text{ } \img alt="rabbit icon" data-bbox="815 575 855 615" \end{bmatrix}$$

$$F_{k+1} = aF_k + bR_k, \text{ and}$$

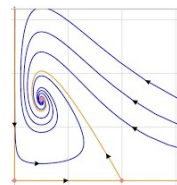
$$R_{k+1} = -rF_k + cR_k, \text{ where } a, b, c, r \in \mathbb{R}.$$

Note: r is rate at which the rabbits are eaten by foxes,
and a, c are reproductive/death rates.

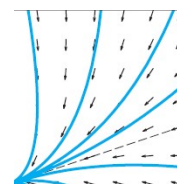
Given this scenario, we set up the vector and matrix: $\vec{x}_k = \begin{bmatrix} F_k \\ R_k \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} a & b \\ -r & c \end{bmatrix}$.

Long term behavior? Note that as $k \rightarrow \infty \dots$

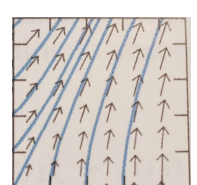
- ◆ F_k and R_k may approach some constant nonzero values,
- ◆ F_k and R_k may oscillate around some constant nonzero values,
- ◆ F_k and R_k may both approach zero,
- ◆ F_k and R_k may both increase without bound.



approaches constant



approaches zero

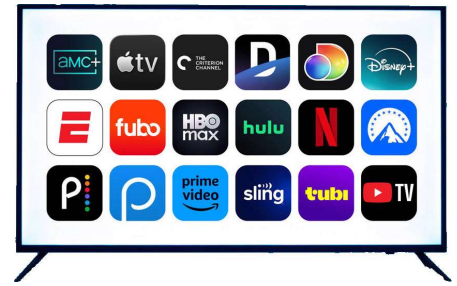


both increase

Assuming the matrix is diagonalizable, below we'll see a method by which we can discover if the populations go extinct, explode, or approach some constant nonzero values.

Example: A streaming service tracks how users divide their time between movies (M) and TV shows (T). Each month, users may shift their viewing habits:

- ♦ 70% of users who primarily watch movies continue primarily watching movies the next month, while 30% switch to mostly TV shows.
- ♦ 10% of users who primarily watch TV shows switch to movies, while 90% continue watching TV shows.



Let M_k and T_k represent the proportion of users primarily watching movies or TV shows (respectively) in month k .

The system evolves according to:

$$\begin{aligned} M_{k+1} &= 0.7M_k + 0.1T_k \\ T_{k+1} &= 0.3M_k + 0.9T_k \end{aligned}$$

Assuming the total number of users remains constant, use diagonalization to find the long-term distribution of users between movies and TV shows.

Solution: Let $\mathbf{A} = \begin{bmatrix} \frac{7}{10} & \frac{1}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix}$ and $\vec{s}_k = \begin{bmatrix} M_k \\ T_k \end{bmatrix}$.

We wish to calculate $\lim_{k \rightarrow \infty} \mathbf{A}^k \vec{s}_0$.

This seems rather difficult, but maybe easier after diagonalization?

Must use: $|\mathbf{A} - \lambda \mathbf{I}| = 0$,

$$\begin{vmatrix} \frac{7}{10} - \lambda & \frac{1}{10} \\ \frac{3}{10} & \frac{9}{10} - \lambda \end{vmatrix} = 0.$$

$$\frac{1}{100} \begin{vmatrix} 7 - 10\lambda & 1 \\ 3 & 9 - 10\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 7 - 10\lambda & 1 \\ 3 & 9 - 10\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7 - 10\lambda)(9 - 10\lambda) - 3 = 100\lambda^2 - 160\lambda + 60 = 20(5\lambda - 3)(\lambda - 1), \text{ so } \lambda = 1 \text{ or } \frac{3}{5}.$$

So, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{bmatrix}$.

Must solve: $(\mathbf{A} - \lambda \mathbf{I})\vec{v} = \vec{0}$.

$$\lambda = 1 : \begin{bmatrix} \frac{7}{10} - 1 & \frac{1}{10} \\ \frac{3}{10} & \frac{9}{10} - 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{3}{10} & \frac{1}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 1 \end{bmatrix}, \text{ so set } y = s, x = \frac{s}{3}.$$

Thus, $(x, y) = (\frac{s}{3}, s) = (1, 3)$, when $s = 3$. Thus, $\vec{v}_1 = (1, 3)$.

$$\lambda = \frac{3}{5} : \begin{bmatrix} \frac{7}{10} - \frac{3}{5} & \frac{1}{10} \\ \frac{3}{10} & \frac{9}{10} - \frac{3}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{3}{10} & \frac{3}{10} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix}, \text{ so set } y = s, x = -s.$$

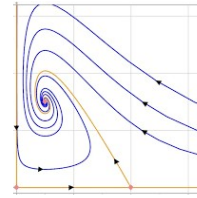
Thus, $(x,y) = (-s,s) = (-1,1)$, when $s = 1$. Thus, $\vec{v}_2 = (-1,1)$.

$$\text{So, } \mathbf{P} = [\vec{v}_1, \vec{v}_2] = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \text{ and } \mathbf{P}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \text{ thus } \mathbf{A}^k = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (\frac{3}{5})^k \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}.$$

$$\begin{aligned} \text{As } k \rightarrow \infty, \text{ we have } \mathbf{A}^\infty &= \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}. \end{aligned}$$

$$\text{So, } \vec{s}_\infty = \mathbf{A}^\infty \vec{s}_0 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} M_0 \\ T_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(M_0 + T_0) \\ \frac{3}{4}(M_0 + T_0) \end{bmatrix}.$$

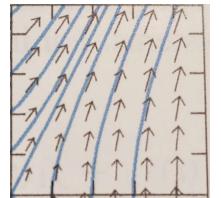
In other words, eventually the number of movie watchers will be 25% of all viewers, and TV watchers will be 34% of viewers.



Note: if $\mathbf{A}$ had been $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b > 1$, then $\mathbf{A}^\infty = \lim_{k \rightarrow \infty} \mathbf{A}^k$ would have been

$$\lim_{k \rightarrow \infty} \mathbf{P} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \mathbf{P}^{-1} = \mathbf{P} \begin{bmatrix} \infty & 0 \\ 0 & \infty \end{bmatrix} \mathbf{P}^{-1}. \text{ And upon multiplying, we'd have found:}$$

$$\vec{s}_\infty = \mathbf{A}^\infty \vec{s}_0 = \begin{bmatrix} \infty \\ \infty \end{bmatrix}, \text{ viewers of each would increase without bound.}$$



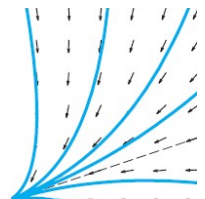
This is impossible since the percentages must add to 1 (but this would work for the predator prey model).

Similarly, if $a, b < 1$, then $\mathbf{A}^\infty = \lim_{k \rightarrow \infty} \mathbf{A}^k$ would have been

$$\lim_{k \rightarrow \infty} \mathbf{P} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \mathbf{P}^{-1} = \mathbf{P} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{P}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

And upon multiplying, we would have found: $\vec{s}_\infty = \mathbf{A}^\infty \vec{s}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Viewers of each would go to zero percent.

This is impossible since the percentages must add to 1 (but this would work for the predator prey model).

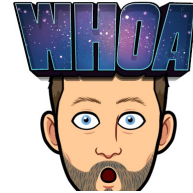


Cayley-Hamilton Theorem

“Every square matrix satisfies its own characteristic equation.”

If $\mathbf{A}$ has the characteristic polynomial: $p(\lambda) = (-1)^3 \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0$, then ...

$$p(\mathbf{A}) = -\mathbf{A}^3 + c_2 \mathbf{A}^2 + c_1 \mathbf{A} + c_0 \mathbf{I} = \mathbf{0}.$$



Exercises 6.3

What did we learn?

- ♦ Transition Matrices
- ♦ Predator-Prey Models
- ♦ Cayley-Hamilton Theorem



Prepared by Dr. Jodin Morey.

Materials for Other Courses Found at **MathTalker.org**