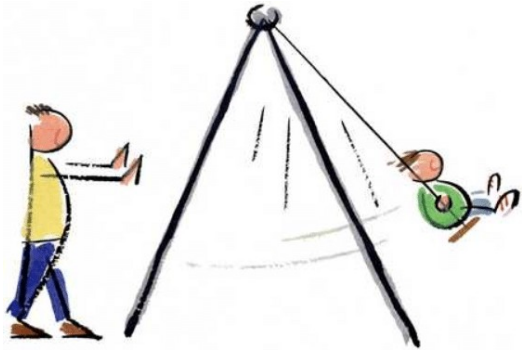


# Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

## .6: Forced Oscillations and Resonance



Forced Swing Oscillations



Forced Yo-Yo

### Forced oscillation with damping:

Equation:  $mx'' + cx' + kx = F(t)$ , where  $F(t) = F_0 \cos \omega t$ , or  $F(t) = F_0 \sin \omega t$ .

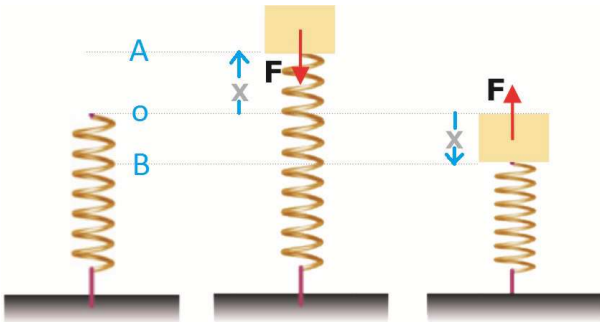
$F_0$  represents the amplitude of the forced oscillation, and  $\omega$  is the circular frequency of the external force  $F(t)$ . Distinguish this from  $\omega_0$  and  $\omega_1$  which we learned about earlier being the undamped and damped circular frequencies of our non-forced system.

### Forced oscillation without damping:

Equation:  $mx'' + kx = F(t)$ , where  $F(t) = F_0 \cos \omega t$ , or  $F(t) = F_0 \sin \omega t$ .

We now know how to calculate the complementary homogeneous solution:

$x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$ , where  $\omega_0 = \sqrt{\frac{k}{m}}$ , is the natural (undamped) frequency of the system. The resulting motion for this forced oscillation is a superposition of the oscillations  $\omega_0$  and  $\omega$ .



Spring Opposing Displacement

### Let's determine how the external force $F_0 \cos \omega t$ effects the Amplitude of the solution

A spring opposes static displacement, so the contribution of  $F_0$  to the solution's amplitude is altered by the spring constant  $k$ :

**Static Displacement of Spring:**  $x_{sd} = \frac{F_0}{k}$  results from constant force with amplitude  $F_0$ .

Also, just like when pushing a child on a swing, the more the forced oscillation  $\omega$  (the push) is in sync with the system's oscillation  $\omega_0$  (the child's swinging), the more amplification (the higher the child will swing). So:

**Amplification Factor:**  $\rho = \frac{1}{\left|1 - \frac{\omega^2}{\omega_0^2}\right|}$ .

**Resonance:** Is the fact that  $\rho$  can be arbitrarily large, when  $\omega$  is near  $\omega_0$ .

The resulting contribution to the solution's amplitude is  $\rho \frac{F_0}{k}$ , so the solution takes the form:  $x(t) = x_c + x_p = C \cos(\omega_0 t - \alpha) \pm \rho \frac{F_0}{k} \cos \omega t$ .

Observe that there is no limit on the amount of amplitude we can get for our

particular solution from  $\rho \frac{F_0}{k}$ .

## Modeling Mechanical Systems:

**Kinetic Energy:**  $T = \frac{1}{2}m(x')^2 = \frac{1}{2}mv^2 = \int_0^v m\bar{v}d\bar{v}$ , for movement of a mass  $m$  with velocity  $v$ . While  $T = \frac{1}{2}I\omega^2$  for rotation of object w/

"momentum of inertia"  $I$ , and angular velocity  $\omega$ .

**Potential Energy:**  $V = \frac{1}{2}kx^2 = \int_0^x k\bar{x}d\bar{x}$ , for a spring constant  $k$  displaced a distance  $x$ .

$V = mgh$ , gravitational potential energy of mass  $m$  at height  $h$  above reference.

## Damped Forced Oscillations: $mx'' + cx' + kx = F(t)$

Situations:  $c > c_{cr}$ ,  $c = c_{cr}$ , or  $c < c_{cr}$ , where  $c_{cr} = \sqrt{4km}$ .

Recall, the gen. solution takes the form:  $x = x_c + x_p$ .

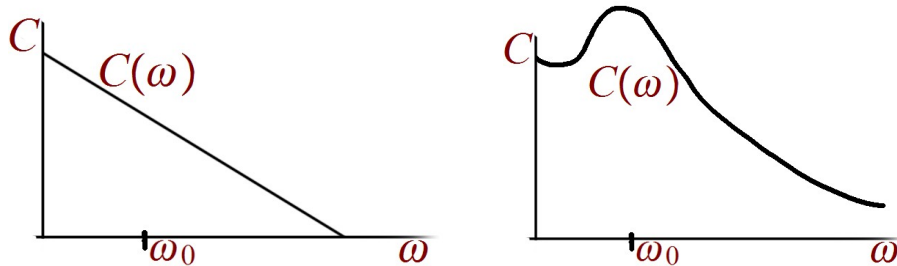
The associated complementary solution,  $x_c(t)$  is a **transient solution** (notationally  $x_{tr}$ ), in that it goes to zero as  $t \rightarrow +\infty$ . Over time, this leaves only the particular solution  $x_p(t)$ , which we refer to as the **steady periodic oscillation** (notationally  $x_{sp}$ ).

**Practical Resonance:** Unlike the unbounded resonance of the undamped oscillations, the resonance of a damped oscillation (in the real world) has a finite amplitude. Similar to critical damping  $c_{cr}$  above, we have the concept of resonant damping  $c_r = \sqrt{2km}$ . So, If you calculate:

$x_{sp}(t) = A \cos \omega t + B \sin \omega t = C(\omega) \cos(\omega t - \alpha)$  :

♦ For  $c \geq c_r = \sqrt{2km}$ ,  $C(\omega)$  is a steadily decreasing function of  $\omega$  (not of time!!).

♦ For  $c < c_r = \sqrt{2km}$ ,  $C(\omega)$  has a local maximum value (*Practical Resonance*), when  $\omega$  is near  $\omega_0$ , then  $C(\omega)$  approaches zero as  $\omega \rightarrow +\infty$ .



Therefore, underdamped systems with forced oscillation typically have amplitudes:

- ♦  $C(\omega)$  large if  $\omega$  is close to the critical resonance frequency ( $\omega_0$ ).
- ♦  $C(\omega)$  close to  $\frac{F_0}{k}$  (Static Displacement of Spring), if  $\omega$  is small.
- ♦  $C(\omega)$  very small if  $\omega$  is very large.

**Videos of Disastrous Resonance:** Collapsing Bridge from wind.

Shattering Glass from sound waves.

**Problem: #10** Given the equation  $x'' + 3x' + 3x = 8 \cos 10t + 6 \sin 10t$ , find the steady periodic solution:

$x_{sp}(t) = C \cos(\omega t - \alpha)$ .

$$r^2 + 3r + 3 \Rightarrow r = \frac{-3 \pm \sqrt{9-12}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}$$

$$x_c = e^{-\frac{3}{2}t} \left( c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t \right).$$

Forming Trial Solution:

$$A \cos 10t + B \sin 10t$$

Comparing this to the complementary solution reveals no linear dependence, so...

$$x_{trial} = A \cos 10t + B \sin 10t$$

$$x'_{trial} = -10A \sin 10t + 10B \cos 10t, \quad x''_{trial} = -100A \cos 10t - 100 \sin B10t$$

$$\begin{aligned} x''_{trial} + 3x'_{trial} + 3x_{trial} &= (-100A \cos 10t - 100B \sin 10t) + 3(-10A \sin 10t + 10B \cos 10t) + 3(A \cos 10t + B \sin 10t) \\ &= (-100A \cos 10t - 100B \sin 10t) + (-30A \sin 10t + 30B \cos 10t) + (3A \cos 10t + 3B \sin 10t) \\ &= (30B - 100A + 3A) \cos 10t + (3B - 100B - 30A) \sin 10t \\ &= (30B - 97A) \cos 10t + (-97B - 30A) \sin 10t \\ &\stackrel{?}{=} 8 \cos 10t + 6 \sin 10t. \end{aligned}$$

Comparing sides of the equation:

$$30B - 97A = 8, \text{ and } -97B - 30A = 6$$

$$B = \frac{8}{30} + \frac{97}{30}A, \quad -97\left(\frac{4}{15} + \frac{97}{30}A\right) - 30A = -\frac{10309}{30}A - \frac{388}{15} = 6,$$

$$A = -\frac{30(6 + \frac{388}{15})}{10309} = -\frac{956}{10309}.$$

$$B = \frac{8}{30} + \frac{97}{30}\left(-\frac{956}{10309}\right) = -\frac{342}{10309}.$$

$$x_{sp}(t) = -\frac{956}{10309} \cos 10t - \frac{342}{10309} \sin 10t \quad \text{Are we done?}$$

"Find the steady periodic solution:  $x_{sp}(t) = C \cos(\omega t - \alpha)$ ."

$$C = \sqrt{\left(-\frac{956}{10309}\right)^2 + \left(-\frac{342}{10309}\right)^2} = \sqrt{\left(\frac{913936}{106275481}\right) + \left(\frac{116964}{106275481}\right)} = \sqrt{\frac{100}{10309}} = \frac{10\sqrt{61}}{793}.$$

$$x_{sp}(t) = \frac{10\sqrt{61}}{793} \cos(10t - \alpha).$$

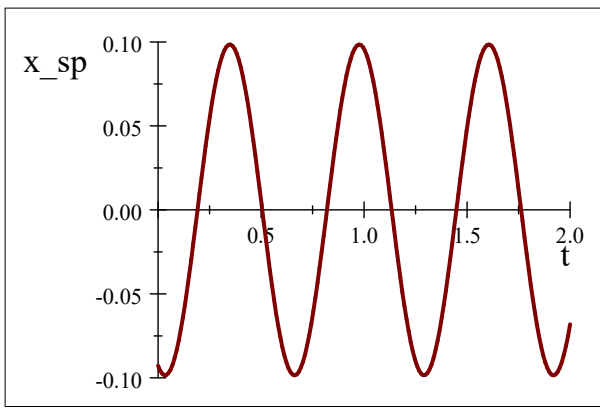
$$\text{where } \alpha = (???) + \tan^{-1} \frac{B}{A}$$

$$\frac{B}{A} = \frac{-\frac{342}{10309}}{-\frac{956}{10309}} = \frac{171}{478}$$

$$\text{Recall: } \alpha = \begin{cases} \tan^{-1} \frac{B}{A} & \text{if } A, B > 0 \text{ (1st quadrant),} \\ \pi + \tan^{-1} \frac{B}{A} & \text{if } A < 0 \text{ (2nd/3rd quadrant),} \\ 2\pi + \tan^{-1} \frac{B}{A} & \text{if } A > 0, B < 0 \text{ (4th quadrant).} \end{cases}$$

$$\alpha = \pi + \tan^{-1} \frac{171}{478} \approx 3.4851. \quad \text{(third quadrant angle)}$$

$$x_{sp}(t) \approx \frac{10\sqrt{61}}{793} \cos(10t - 3.485).$$



$$\frac{10\sqrt{61}}{793} \cos(10t - 3.485)$$

**Problem: #13** Find and plot the steady periodic solution  $x_{sp}(t) = C \cos(\omega t - \alpha)$  of the differential equation  $x'' + 2x' + 26x = 600 \cos 10t$ , with initial conditions  $x(0) = 10$ , and  $x'(0) = 0$ . Also, find and plot the actual solution:  $x(t) = x_{tr}(t) + x_{sp}(t) = x_c(t) + x_p(t)$ .

$$r^2 + 2r + 26 = 0, \quad r = \frac{-2 \pm \sqrt{4 - 104}}{2} = -1 \pm 5i.$$

$$x_c(t) = x_{tr}(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)$$

$$x_0 = x_{trial} = A \cos 10t + B \sin 10t$$

$$x'_{trial} = -10A \sin 10t + 10B \cos 10t$$

$$x''_{trial} = -100A \cos 10t - 100B \sin 10t$$

$$x''_{trial} + 2x'_{trial} + 26x_{trial} =$$

$$\begin{aligned} & (-100A \cos 10t - 100B \sin 10t) + 2(-10A \sin 10t + 10B \cos 10t) + 26(A \cos 10t + B \sin 10t) \\ &= (-100A \cos 10t - 100B \sin 10t) + (-20A \sin 10t + 20B \cos 10t) + (26A \cos 10t + 26B \sin 10t) \\ &= (20B - 100A + 26A) \cos 10t + (26B - 100B - 20A) \sin 10t \\ &= (20B - 74A) \cos 10t + (-74B - 20A) \sin 10t \\ &\stackrel{?}{=} 600 \cos 10t. \end{aligned}$$

**Comparing sides of the equation:**

$$20B - 74A = 600, \quad -74B - 20A = 0.$$

$$B = -\frac{20}{74}A, \quad 20\left(-\frac{20}{74}A\right) - 74A = -\frac{2938}{37}A = 600, \quad A = -\frac{600 \cdot 37}{2938} = -\frac{11100}{1469}$$

$$B = -\frac{20}{74}\left(-\frac{11100}{1469}\right) = \frac{3000}{1469}.$$

$$\text{So, } x_{sp}(t) = -\frac{11100}{1469} \cos 10t + \frac{3000}{1469} \sin 10t. \quad \text{Now what?}$$

**Putting it in the right form:**

$$C = \sqrt{\left(-\frac{11100}{1469}\right)^2 + \left(\frac{3000}{1469}\right)^2} = \frac{300}{\sqrt{1469}}.$$

$$x_{sp}(t) = \frac{300}{\sqrt{1469}} \cos(10t - \alpha).$$

Determining  $\alpha$  :

$$\frac{B}{A} = \frac{\frac{3000}{1469}}{-\frac{11100}{1469}} = -\frac{10}{37}$$

$$\alpha = \pi + \tan^{-1} \frac{10}{-37} \approx 2.88 \qquad \text{(second quadrant angle)}$$

$$\text{So, } x_{sp}(t) \approx \frac{300}{\sqrt{1469}} \cos(10t - 2.88).$$

$$\textbf{Gen. solution: } x(t) = x_{tr}(t) + x_{sp}(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t) + \frac{300}{\sqrt{1469}} \cos(10t - 2.8776).$$

Are we done?

"Find the actual solutions  $x(t) = x_{sp}(t) + x_{tr}(t)$  that satisfies the given initial conditions.

$$x(0) = 10, \qquad x'(0) = 0."$$

To keep the accuracy, and not deal with decimals, I revert back to the previous version for this...

$$x(t) = x_{tr} + x_{sp}(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t) - \frac{11100}{1469} \cos 10t + \frac{3000}{1469} \sin 10t$$

$$x'(t) = [e^{-t}(5c_2 \cos 5t - 5c_1 \sin 5t) - e^{-t}(c_1 \cos 5t + c_2 \sin 5t)] + \frac{111000}{1469} \sin 10t + \frac{30000}{1469} \cos 10t$$

$$x'(0) = 0 = [e^0(5c_2 \cos 0 - 5c_1 \sin 0) - e^0(c_1 \cos 0 + c_2 \sin 0)] + \frac{111000}{1469} \sin 0 + \frac{30000}{1469} \cos 0$$

$$= [(5c_2) - (c_1)] + \frac{30000}{1469}$$

$$x(0) = 10 = e^0(c_1 \cos 0 + c_2 \sin 0) - \frac{11100}{1469} \cos 0 + \frac{3000}{1469} \sin 0 = c_1 - \frac{11100}{1469}$$

$$c_1 = \frac{11100}{1469} + 10 = \frac{25790}{1469}, \qquad c_2 = \frac{1}{5} c_1 - \frac{1}{5} \frac{30000}{1469} = \frac{1}{5} \frac{25790}{1469} - \frac{1}{5} \frac{30000}{1469} = -\frac{842}{1469}.$$

$$\text{Therefore, } x(t) = e^{-t} \Big( \frac{25790}{1469} \cos 5t - \frac{842}{1469} \sin 5t \Big) + \frac{300}{\sqrt{1469}} \cos(10t - 2.88).$$

We done?

We could convert the complementary solution into the form:  $x_{tr}(t) = C \cos(\omega t - \alpha)$ .

$$x_{tr}(t) = \frac{1}{1469} e^{-t}(25790 \cos 5t - 842 \sin 5t)$$

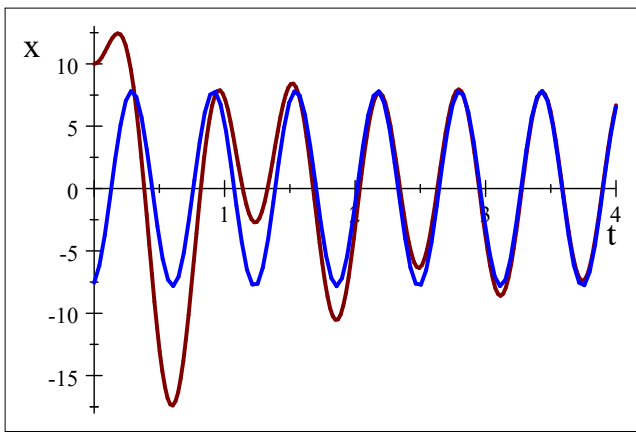
$$C = \sqrt{25790^2 + (-842)^2} = 2\sqrt{166458266}$$

$$x_{tr}(t) = \frac{2\sqrt{166458266}}{1469} e^{-t} \cos(5t - \beta)$$

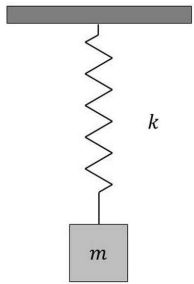
$$\frac{B}{A} = \frac{-842}{25790} = -\frac{421}{12895}$$

$$\beta = 2\pi + \tan^{-1} \frac{-421}{12895} \approx 6.2505. \qquad \text{(fourth quadrant angle)}$$

$$x(t) \approx \frac{2\sqrt{166458266}}{1469} e^{-t} \cos(5t - 6.2505) + \frac{300}{\sqrt{1469}} \cos(10t - 2.8776)$$



$x$  (red), and  $x_{sp}$  (blue)



**Problem 19:** A mass weighing 100 pounds (mass  $m = 3.125$  slugs in *fps* units) is attached to the end of a spring that is stretched 1 inch, by a force of 100 pounds (duh!). Another force ( $F_0 \cos \omega t$ ) acts on the mass. At what frequency in hertz (cycles/sec) will resonant oscillations occur? Neglect damping.

Resonant frequency is  $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{3.125}}$ .

$m = 3.125$  slugs (*fps*), and  $k = \frac{\text{Force}}{\text{Distance}} = \frac{100 \text{ lbs}}{\frac{1}{12} \text{ ft}} = 1200 \text{ lb/ft}$ .

So the resonant frequency is  $\omega_0 = \sqrt{\frac{1200}{3.125}}$  (units?)

$$= \sqrt{384} \frac{\text{rad}}{\text{sec}} \approx 19.6 \frac{\text{rad}}{\text{sec}}.$$

$$\# \text{ of Hz} = \frac{\omega_0}{2\pi} = \frac{\sqrt{384}}{2\pi} \approx 3.12 \text{ Hz}.$$

So, resonance will occur when  $\frac{\omega}{2\pi} \approx \frac{\omega_0}{2\pi} = 3.12 \text{ Hz}$ .

If  $c = 12$ , and  $\omega = 90$ , what behavior should we expect from the amplitude?

Resonant damping:  $c_r = \sqrt{2km} = \sqrt{2 \cdot 1200 \cdot 3.125} \approx 86.6$ .

Recall  $\omega_0 \approx 19.6$ , and "for  $c < \sqrt{2km}$ ,  $C(\omega)$  reaches a maximum value (*Practical Resonance*),

at some value of  $\omega < \omega_0$ , then it approaches zero as  $\omega \rightarrow +\infty$ ."

We should expect the amplitude to be less than the practical resonance.

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If  $c = 200$  and  $\omega$  is very large, what behavior should we expect from the amplitude?

**Recall:**     ♦ For  $c \geq \sqrt{2km}$ ,  $C(\omega)$  is a steadily decreasing function of  $\omega$ .

So the amplitude should be small.

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If  $c = 12$ , and  $\omega < 20$ , what behavior should we expect from the amplitude?

The amplitude may be at its peak (practical resonance), or at least not very small.

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