

Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ◆ n th-order characteristic equation
- ◆ Differential Operators L acting on $f(x)$
- ◆ Repeated Roots
- ◆ Complex Roots



5.5 - Non-Homogeneous DEQs & Undetermined Coefficients

Non-homogeneous DEQs are of the form: $Ly = f(x)$, where L is a differential operator explained in 5.3.

Linear example w/constant coefficients: $y^{(7)} + 2y^{(3)} + 3y'' + 4y' + 5y = f(x)$. (*)

How to solve them?

Earlier we learned that general sols have the form $y = y_c + y_p$, where y_c is the complementary soln obtained from the characteristic equation, and y_p is *any* particular soln. But how do we find a y_p ?

Undetermined Coefficients, the Justification

To solve a linear DEQ w/constant coefficients like (*) above, we need some function y such that when we take various derivatives ($y^{(7)}, y^{(3)}, y'', y', y$), then multiply them by some constants ($1y^{(7)}, 2y^{(3)}, 3y'', 4y', 5y$), and add them together, we get $f(x)$. So if we limit the type of expression $f(x)$ can be, we might come up with a good guess for y .

Polynomial: If we assume $f(x)$ is a polynomial, recall that the derivative of a polynomial $y(x)$ is a polynomial. So a reasonable guess y_i for a particular soln to substitute into the LHS of the DEQ would be a polynomial $y_i := A_1 + A_2x + \dots + A_{n-1}x^n$. The A_j are yet-to-be-determined coefficients. Substituting y_i into the LHS and taking derivatives as necessary, we could then compare the two sides of the equation to determine the *undetermined coefficients* A_i .

Exponential: Similarly, if $f(x)$ is an exponential functions (e.g., $7e^{5x}$), note that the derivative of an exponential $y(x)$ is also an exponential ($(e^{5x})' = 5e^{5x}$). So Ae^{5x} would be a reasonable guess y_i for a particular soln to substitute into the LHS of the DEQ. The A is a yet-to-be-determined coefficient.

Trigonometric: Similarly, if $f(x)$ is a sine or cosine function (e.g., $7 \cos 3x$), note that the derivative of $\sin/\cos$ is also $\cos/\sin$. So $A \sin 3x + B \cos 3x$ would be a reasonable guess y_i for a particular soln to substitute into the LHS of the DEQ. The A, B are yet-to-be-determined coefficients.

Even better, we can merge these three facts into a procedure (seen below) which allows for $f(x)$ to be of various forms that combine these types of functions.

Linear Independence

Before we write down the procedure, there's still one difficulty.
The processes laid out above may result in a y_i that has terms that are linearly dependent with terms in y_c .

For example from: $y''' - 3ry'' + 3r^2y' - r^3y = (2x - 3)e^{rx}$ the above process would result in $y_c = c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx}$, and $y_i = (A + Bx)e^{rx} = Ae^{rx} + Bxe^{rx}$. So what's wrong with this?

First, observe that the terms in y_i are linearly dependent with terms in y_c
Note: Ae^{rx} and kc_1e^{rx} differ by a constant. Also Bxe^{rx} , kc_2xe^{rx} differ by a constant.

In other words, y_i just consists of sols already covered by our y_c .

But y_i consisting of sols to the associated homogeneous DEQ means that
substituting them into our non-homogeneous DEQ results in $0 = (2x - 3)e^{rx}$.
So they are *not* a sols to the non-homogeneous DEQ!



This also doesn't give us the opportunity to solve for the undetermined coefficients A, B in y_i .



So how do we alter y_i to produce the sols we're looking for?
Multiply the dependent terms of y_i by x , until they're linearly independent from the terms in y_c !

Further Justification

To explore this, we first rewrite our DEQ above as: $(D - r)^3y = 2xe^{rx} - 3e^{rx}$.

Then we recall something from section 5.3, that is $(D - r)^k[u(x)e^{rx}] = D^k(u(x))e^{rx}$.

So if we multiply both sides of our DEQ by $(D - r)^2$, we have:
$$(D - r)^5y = (D - r)^2(2xe^{rx} - 3e^{rx}) = (D - r)^2(2xe^{rx}) - (D - r)^2(3e^{rx})$$
$$= D^2(2x)e^{rx} - D^2(3)e^{rx} = 0 \cdot e^{rx} - 0 \cdot e^{rx} = 0.$$

In other words, any soln y which satisfies our original non-homogeneous DEQ,
 $((D - r)^3y = (2x - 3)e^{rx})$ also satisfies $(D - r)^5y = 0$.

Observe from our previous analysis that sols to this DEQ can take the form:
 $y(x) = c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx} + Ax^3e^{rx} + Bx^4e^{rx}$, where I've suggestively chosen notation for the constant coefficients.

In other words, if I multiply y_i by x^3 , these are likely to produce sols to my non-homogeneous DEQ.

Undetermined Coefficients, the Method

The method of **Undetermined Coefficients** assumes f is of the form: $f(x) = Ax^k e^{rx} \cos(sx)$ or $Ax^k e^{rx} \sin(sx)$, where $k, r, s \geq 0$.
(or $f(x)$ can consist of several terms of this form added together)

Steps to solving...

◆ Determine **complementary soln** $y_c = c_1 y_1 + c_2 y_2$. (from the homogeneous DEQ, where $f(x) = 0$)

Example, for $y'' + 2y' = f(x)$, we calculate: $y_c = c_1 + c_2 e^{-2x}$.

◆ Define a **pre-trial soln**: $y_i := p_1(x) + \dots + p_n(x)$.

Example, if $f(x) = \sin x + 7xe^{-2x}$, then $y_i = A \sin x + B \cos x + (C + Dx)e^{-2x}$. (* *)

Find the pre-trial soln in three steps. For each term in $f(x)$:

◆ **Trig-Step**: If there's $\sin(sx)$ or $\cos(sx)$ in the term, write: $\sin(sx) + \cos(sx)$.

◆ **Exponential-Step**: If there's an exponential e^{rx} , multiply what you have by e^{rx} . Example: $e^{rx} \sin(sx) + e^{rx} \cos(sx)$.

◆ **Power-Step**: Finally, determine k (the power of x). (Note that you may have $k = 0$.)

Then, multiply each term of what you have by $(A + Bx + Cx^2 + \dots + Lx^k)$,
with different constants for each term. If $k = 0$, then you just multiply by A .

◆ Next, we need for the terms of our **trial soln** to be linearly independent from our **complementary soln**.

So, for each term p_i of our **pre-trial soln**: $y_i = p_1(x) + \dots + p_n(x)$, determine the smallest power s_i of x ,
such that $x^{s_i} p_i$ isn't a constant multiple of any term in our complementary soln: $y_c = c_1 y_1 + c_2 y_2$.
(we're removing duplicates to achieve independence of the two sets of sols).

Putting it together we have a **trial soln**: $y_{trial} = x^{s_1} p_1 + \dots + x^{s_n} p_n$.

(continuing w/our example (* *) above: $y_{trial} = A \sin x + B \cos x + Cx^2 e^{-2x} + Dx^3 e^{-2x}$).

◆ Substitute y_{trial} into $Ly = f(x)$ (taking derivatives as necessary), and determine the coefficients $(A, B, \dots)$ by comparing the two sides of the equation. The result we label y_p (our particular soln).

◆ **General Solution**: $y = y_c + y_p$. (combination of complementary & particular soln)

Example: Solve the initial value problem: $y'' + y = \cos x$; $y(0) = 1$, $y'(0) = -1$.

Complementary soln: $r^2 + 1 = 0$, $r = \pm i$, $\Rightarrow e^{ix} = \cos x + i \sin x$.

So, $y_c = c_1 \cos x + c_2 \sin x$;

Finding a pre-trial soln y_i :

$$y_i = A \cos x + B \sin x \quad \text{Linearly independent from } y_c?$$

$$y_{trial} = x(A \cos x + B \sin x)$$

Differentiating y_{trial} to plug back into our equation...

$$y'_{trial} = (-A \sin x + B \cos x)x + (A \cos x + B \sin x) = (Bx + A) \cos x + (-Ax + B) \sin x$$

$$\begin{aligned} y''_{trial} &= B \cos x - (Bx + A) \sin x + (-A) \sin x + (-Ax + B) \cos x \\ &= (-Ax + 2B) \cos x + (-Bx - 2A) \sin x \end{aligned}$$

Plugging them back into $y'' + y = \cos x$, we get:

$$\begin{aligned} [(-Ax + 2B) \cos x + (-Bx - 2A) \sin x] + [Ax \cos x + Bx \sin x] \\ = 2B \cos x - 2A \sin x = \cos x. \end{aligned}$$

Determining our coefficients...

$$2B = 1, \text{ and } -2A = 0. \quad A = 0 \text{ and } B = \frac{1}{2}.$$

$$y_p = \frac{1}{2}x \sin x.$$

Therefore, our general solution is...

$$y_g = y_c + y_p = c_1 \cos x + c_2 \sin x + \frac{1}{2}x \sin x. \quad \text{And...}$$

Using the initial conditions to solve for c_1 and c_2 ...

$$1 = c_1 \cos 0 - 1 \sin 0 + \frac{1}{2}(0) \sin 0 = c_1.$$

$$y' = -c_1 \sin x + c_2 \cos x + \frac{1}{2} \sin x + \frac{1}{2}x \cos x$$

$$-1 = -c_1 \sin 0 + c_2 \cos 0 + \frac{1}{2} \sin 0 + \frac{1}{2}(0) \cos 0 = c_2.$$

Plugging these into y ...

$$y(x) = \cos x - \sin x + \frac{1}{2}x \sin x.$$

□

Here's a video explanation from Khan Academy:

[khanacademy.org/math/differential-equations/second-order-differential-equations/undetermined-coefficients/v/undetermined-coefficients-1](https://www.khanacademy.org/math/differential-equations/second-order-differential-equations/undetermined-coefficients/v/undetermined-coefficients-1)

But what if $f(x)$ isn't in the form required by Undetermined Coefficients? ($Ax^k e^{rx} \cos(sx)$ or $Ax^k e^{rx} \sin(sx)$)

Or what if our linear DEQ has variable coefficients?

Variation of Parameters, the Justification

So how do we form y_p ? We saw w/undetermined coefficients that sometimes the particular sols to our DEQ are similar to the complimentary sols (y_1, y_2 , etc.) in $y_c = c_1y_1 + c_2y_2 + \dots$, but multiplied by some power of x .

What if we made the assumption that something similar happens for more complicated $f(x)$; that our particular soln takes the form: $y_p = u_1y_1 + u_2y_2 + \dots$, where u_i are unknown functions of x .

Below, we work with a 2nd-order DEQ, but the same process works for n th order DEQs!
Also, unlike undetermined coefficients, our DEQ can have variable coefficients.
This represents much more flexibility in the types of DEQs we can solve!

If: $y'' + P(x)y' + Q(x)y = f(x)$, with $y_c = c_1y_1 + c_2y_2$, we write a guess to the particular soln as: $y_p := u_1y_1 + u_2y_2$.

If we were to substitute this into our DEQ, there would be two unknown functions u_1, u_2 , but only one eqn (one constraint). That is, our DEQ. Generally, one would need two eqns to pin down both u_1, u_2 .

We could write down a new constraint, but how would we know if it was correct? That's easy!
If the resulting constraint allows us to find some equations u_1, u_2 , and they solve our DEQ, then the constraint was correct.
It turns out the desired new constraint is $u_1'y_1 + u_2'y_2 = 0$.

So let's substitute y_p into our DEQ. We'll use our new constraint along the way to simplify things.
This will simplify our given DEQ, and we'll end up with two constraints.
These will allow us to solve this type of DEQ.

Note: $y_p' = (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2)$.

Applying our new constraint, this simplifies to: $y_p' = u_1y_1' + u_2y_2'$.

Taking another derivative: $y_p'' = (u_1'y_1' + u_2'y_2') + (u_1y_1'' + u_2y_2'')$. (* * *)

Recall that both y_1, y_2 satisfy the *homogeneous* DEQ: $y_i'' + Py_i' + Qy_i = 0$. Rearranging: $y_i'' = -Py_i' - Qy_i$.

So substituting this into (* * *):

$$\begin{aligned} y_p'' &= (u_1'y_1' + u_2'y_2') + (u_1(-Py_1' - Qy_1) + u_2(-Py_2' - Qy_2)) \\ &= (u_1'y_1' + u_2'y_2') - P \cdot (u_1y_1' + u_2y_2') - Q \cdot (u_1y_1 + u_2y_2) = (u_1'y_1' + u_2'y_2') - Py_p' - Qy_p. \end{aligned}$$

Substituting this into our DEQ ($y_p'' + P(x)y_p' + Q(x)y_p = f(x)$):
 $[(u_1'y_1' + u_2'y_2') - Py_p' - Qy_p] + Py_p' + Qy_p = u_1'y_1' + u_2'y_2' = f(x)$.

So our two constraints become: $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = f(x)$.
Note, the first is the "new" constraint, and the second is the old constraint (the original DEQ), after simplifying using the new constraint.

Variation of Parameters, the Method

This is for non-homogeneous linear DEQs $Ly = f(x)$. It's not restricted to the form necessary for Undetermined Coefficients.

Steps to solving...

1. Determine the complementary soln: $y_c = c_1y_1(x) + c_2y_2(x)$.
2. Differentiate: y_1, y_2 to get y_1', y_2' .
3. Write down: $u_1'y_1 + u_2'y_2 = 0$, and $u_1'y_1' + u_2'y_2' = f(x)$; where u_1', u_2' are unknown.
4. Solve for u_1' and u_2' (2 eqs, 2 unknowns).
5. Integrate u_1' and u_2' to get u_1 and u_2 . (using zeros as the constants of integration).
6. Particular Soln is: $y_p = u_1y_1 + u_2y_2$.
7. As above, the Gen. Soln is: $y_g = y_c + y_p$.

There is another way to characterize this.

Observe that if we solve for u_1' in the first equation of step 3, we have $u_1' = -\frac{u_2'y_2}{y_1}$.

And substituting this in the second equation: $\left(-\frac{u_2'y_2}{y_1}\right)y_1' + u_2'y_2' = u_2'\left(-\frac{y_2}{y_1}y_1' + y_2'\right) = f(x)$.

And solving for u_2' gives $u_2' = \frac{y_1 \cdot f(x)}{y_1y_2' - y_1'y_2}$.

Substituting this back into the first equation gives: $u_1' = -\frac{\left(\frac{y_1 \cdot f(x)}{y_1y_2' - y_1'y_2}\right)y_2}{y_1} = -\frac{y_2 \cdot f(x)}{y_1y_2' - y_1'y_2}$.

Observe that $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$, where $W(y_1, y_2)$ is the Wronskian.

Therefore, we have $u_1' = -\frac{y_2 \cdot f(x)}{W}$, and $u_2' = \frac{y_1 \cdot f(x)}{W}$.

So, alternatively to steps 2-6 above we have: $y_p = u_1y_1 + u_2y_2 = -y_1 \int \frac{y_2 \cdot f(x)}{W} dx + y_2 \int \frac{y_1 \cdot f(x)}{W} dx$.

Example: Use variation of parameters to solve $y'' + y = \sec x$.

Solving the associated homogeneous DEQ: $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow e^{ix} = \cos x + i \sin x$

$\Rightarrow y = c_1y_1 + c_2y_2$, where $y_1 = \cos x$ and $y_2 = \sin x$.

Now assume a particular solution: $y_p = u_1(x)y_1 + u_2(x)y_2 = u_1(x)\cos x + u_2(x)\sin x$.

Now imposing the 1st constraint: $u_1'y_1 + u_2'y_2 = 0$, gives us: $u_1'(x)\cos x + u_2'(x)\sin x = 0$.

If we were to substitute this into our DEQ, we would find the 2nd constraint: $u_1'(x)y_1' + u_2'(x)y_2' = f(x)$

Or, $-u'_1(x) \sin x + u'_2(x) \cos x = \sec x$.

Solving for u'_1 from the 1st constraint, $u'_1 = -\frac{u'_2 \sin x}{\cos x}$.

Substituting into the 2nd constraint: $-\left(-\frac{u'_2 \sin x}{\cos x}\right) \sin x + u'_2(x) \cos x = \sec x$

Solving for u'_2 : $u'_2\left(\frac{\sin x}{\cos x} \sin x + \cos x\right) = u'_2\left(\frac{\sin^2 x + \cos^2 x}{\cos x}\right) = u'_2 \sec x = \sec x \Rightarrow u'_2 = 1$.

Substituting into the 1st constraint: $u'_1 = -\frac{\sin x}{\cos x}$.

Integrating to find our unknown functions: $u_2 = \int u'_2 dx = \int 1 dx = x$. (we set $C = 0$)

$u_1 = \int u'_1 dx = -\int \frac{\sin x}{\cos x} dx = \ln|\cos x|$.

Therefore: $y_p = \cos(x) \ln|\cos x| + x \sin x$, with the general solution: $y = c_1 \cos x + c_2 \sin x + \cos(x) \ln|\cos x| + x \sin x$. $\square$

Exercises 5.5

What did we learn?

- ♦ Undetermined Coefficients
- ♦ Variation of Parameters



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Materials for Other Courses Found at **MathTalker.org**