

Previous Lecture

♦ n th-order (general) versions of 1st-order thms



5.3: n th-order Homogeneous DEQs w/Constant Coefficients

$$y^{(n)} + d_1 y^{(n-1)} + \dots + d_{n-1} y' + d_n y = 0, \text{ where } d_i \in \mathbb{R} \quad (*)$$

For an n th-order homogeneous DEQ, we know that if we can find n linearly independent sols, we can form a general soln: $y = c_1 y_1 + \dots + c_n y_n$. So let's find some!

Observe that $(e^{rt})^{(n)} = r^n e^{rt}$. From this, we might suspect sols like $y = e^{rt}$ (where $r \in \mathbb{R}$).

Substituting e^{rt} into $(*)$, we get: $(e^{rt})^{(n)} + d_1 (e^{rt})^{(n-1)} + \dots + d_{n-1} (e^{rt})' + d_n (e^{rt})$

$$= r^n e^{rt} + d_1 r^{n-1} e^{rt} + \dots + d_{n-1} r e^{rt} + d_n e^{rt}$$

$$= e^{rt} (r^n + d_1 r^{n-1} + \dots + d_{n-1} r + d_n) = 0.$$

However, e^{rt} is never zero, so we need only find r satisfying the **characteristic equation**:

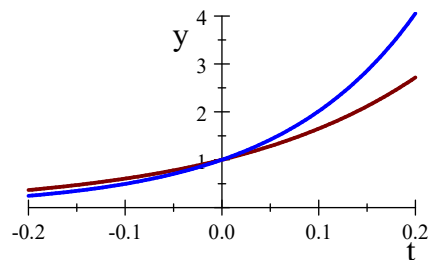
$$r^n + d_1 r^{n-1} + \dots + d_{n-1} r + d_n = 0 \quad (* *).$$

Assuming we find the roots $(r_1, r_2, \dots)$, what's the soln to the DEQ?

Observe that for any $r_1 \neq r_2$, then $e^{r_1 t} \neq k e^{r_2 t}$, for any $k \in \mathbb{R}$.

Therefore, $e^{r_1 t}, e^{r_2 t}$ are independent.

$$W(e^{r_1 t}, e^{r_2 t}) := \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{r_1 t} e^{r_2 t} (r_2 - r_1) \neq 0$$



e^{5t} and e^{7t}

Distinct Real Roots Thm: With *real distinct* roots $r_1, r_2, \dots, r_n$ of the characteristic equation, a general soln is:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}. \text{ And, } \{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\} \text{ is a basis for the soln space to } (*).$$

Differential Operators L acting on $f(x)$: (?!?!?)

We are used to seeing functions f, g as operators that operate on *variables* (x, y) : by raising the variables to some power, adding them in linear combinations, etc. (e.g., $f(x) = 7x^2 + 2x + 7$).

But we can also define operators L that operate on *functions* by taking some derivatives, adding them in linear combinations, etc. (e.g., $L(f(x)) = 7f'' + 2f + 7$).

Identity operator I : $I(f) = f$.

Derivative operator D^i : $D^2(f) = f''$.

So if you have $a_2y'' + a_1y' + a_0y = 0$, we can rewrite it as:

$$(a_2D^2 + a_1D + a_0I)y = L(y) = 0, \text{ where } L := (a_2D^2 + a_1D + a_0I) \text{ is a differential operator.}$$

Having defined a differential operator L , we can write DEQs more succinctly as $Ly = f(x)$.

Repeated Roots

Recall: $y^{(n)} + d_1y^{(n-1)} + \dots + d_{n-1}y' + d_ny = 0$ (*) (homogeneous DEQ)

$$r^n + d_1r^{n-1} + \dots + d_{n-1}r + d_n = 0$$

(***) (characteristic equation)

Repeated Roots (of the characteristic equation): If a root r has multiplicity k (as in $(x - r)^k$), then the corresponding part of the general soln has the form: $(c_1 + c_2x + \dots + c_kx^{k-1})e^{rx}$.

Repeated Roots Proof: Fundamental thm of algebra guarantees $r_1, \dots, r_n$ roots to (**).

If we assume r_1 is our repeated root, then (**) can be rewritten as:

$$(r - r_1)^k(r - r_2)^{n_2} \dots (r - r_j)^{n_j} = 0, \text{ where } k + n_2 + \dots + n_j = n.$$

We need only focus on $(r - r_1)^k$.

The corresponding operator L in (**) is $(D - r_1)^k$, and the requirement on y for these roots is that $(D - r_1)^k y = 0$.

We know e^{r_1x} is a soln, and we suspect other linearly independent versions of this soln like: $y = u(x)e^{r_1x}$.

Substituting this into the requirement, we have:

$$\begin{aligned} &(D - r_1)^k[ue^{r_1x}] \\ &= (D - r_1)^{k-1}(D - r_1)[ue^{r_1x}] \\ &= (D - r_1)^{k-1}(D(ue^{r_1x}) - r_1ue^{r_1x}) \end{aligned}$$

$$= (D - r_1)^{k-1} (e^{r_1 x} Du + u D(e^{r_1 x}) - r_1 u e^{r_1 x}) \quad (\text{product rule})$$

$$= (D - r_1)^{k-1} (e^{r_1 x} Du + r_1 u e^{r_1 x} - r_1 u e^{r_1 x})$$

$$= (D - r_1)^{k-1} e^{r_1 x} Du$$

$\vdots$

$$= e^{r_1 x} D^k u = 0.$$

But we know $e^{r_1 x} \neq 0$, so we need $D^k u = 0$.

If we assume $u(x)$ is a polynomial, the largest order polynomial that would satisfy this is:

$$c_k x^{k-1} + c_{k-1} x^{k-2} + \dots + c_2 x + c_1. \quad (* * *)$$

But any smaller order polynomial would also satisfy the requirement.

Indeed, a basis of functions which would span this function space would be: $\{1, x, \dots, x^{k-1}\}$.

Putting these in a linear combination gives you $(* * *)$.

So the sols $y = u(x)e^{r_1 x}$ would include $e^{r_1 x}, x e^{r_1 x}, \dots, x^{k-2} e^{r_1 x}, x^{k-1} e^{r_1 x}$.

Note that there are k of them, and they are linearly independent.

$$(e^{r_1 x} x^{k_1} = C e^{r_1 x} x^{k_2} \Rightarrow x^{k_1 - k_2} = C, \text{ function on LHS and constant on RHS})$$

Therefore, we can add them in a linear combination to find the full set of sols for r_1 :

$$c_k x^{k-1} e^{r_1 x} + c_{k-1} x^{k-2} e^{r_1 x} + \dots + c_2 x e^{r_1 x} + c_1 e^{r_1 x}. \quad \blacksquare$$

Example: Find the general solution for $y^{(3)} + y'' - y' - y = 0$.

$$r^3 + r^2 - r - 1 = 0.$$

How to solve? Educated guesses for roots to polynomials of this form, having a constant term c are $\pm$ the factors of c . For instance, if you had a polynomial $r^3 + r^2 - r - 6 = 0$, some good educated guesses would be $r \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$ (the factors of 6).

For our problem, observe that $r = 1$ is a root ($r = -1$ works too).

I have to use polynomial division! So dividing by $(r - 1)$: (you can use any method for poly division you know)

$$\frac{r^3 + r^2 - r - 1}{r - 1} = \text{?????} \quad \text{or} \quad r^3 + r^2 - r - 1 = (r - 1) \cdot (\text{?????})$$

$$r^3 + r^2 - r - 1 = (r - 1)r^2 + (2r^2 - r - 1)$$

Further dividing the remainder: $2r^2 - r - 1 = (r - 1)2r + (r - 1) = (r - 1)(2r + 1)$

Therefore, $r^3 + r^2 - r - 1 = (r - 1)(r^2 + 2r + 1) = (r - 1)(r + 1)^2$.

So $r \in \{1, -1\}$ w/multiplicity of two for -1 .

Thus, $y = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}$.

Complex Roots

A happy consequence of restricting our DEQs to *real coefficients* (d_i) is that the characteristic equation also has real coefficients, and thus any complex roots of this characteristic equation come in *conjugate pairs* (this comes from modern/abstract algebra). In other words, if you find a root $3 + 2i$, you're guaranteed to also have the root $3 - 2i$.

But how do we formulate sols to our DEQ when the roots are complex?



Complex Numbers ($z \in \mathbb{C}$):

Euler's Formula: $e^{i\theta} = \cos \theta + i \sin \theta$.

Proof: Let $f(\theta) = e^{-i\theta}(\cos \theta + i \sin \theta)$.

We need to show that $f(\theta) \equiv 1$. (why?)

Differentiating: $f'(\theta) = -ie^{-i\theta}(\cos \theta + i \sin \theta) + e^{-i\theta}(-\sin \theta + i \cos \theta) = 0$.

Since the derivative $f'(\theta)$ is 0, this means that $f(\theta)$ is a constant.

Also note that $f(0) = 1$. Therefore, $f(\theta) = 1$ for all θ .

Therefore, $1 = e^{-i\theta}(\cos \theta + i \sin \theta)$, or equivalently $e^{i\theta} = \cos \theta + i \sin \theta$. ■

So, if $z = 7 + 3i$, then $e^{zt} = \dots$

$$e^{7t+3it} = e^{7t}e^{3it} = e^{7t}(\cos 3t + i \sin 3t) = e^{7t} \cos 3t + ie^{7t} \sin 3t.$$

$e^{7t} \cos 3t$ is called the **real part** of the number e^{zt} , and

$e^{7t} \sin 3t$ is the **imaginary part** (even though $e^{7t} \sin 3t$ is a real number!!).

Complex-Valued Functions:

For $F(t) = f(t) + ig(t)$, (note that the values thees functions take are complex)

we can take the derivative as: $F'(t) = f'(t) + ig'(t)$.

In order for $F(t)$ to be the soln of a homogeneous DEQ ($L[F(t)] = 0$), it is necessary that both the real part and the imaginary part of the function are **also** each sols. In other words: $L[f(t)] = 0$ AND $L[g(t)] = 0$.

Since (in this course) we're interested in only real valued functions, if a characteristic equation has complex conjugate roots $(7 \pm 3i)$, the soln initially looks like the complex function $c_1 e^{7t} (\cos 3t + i \sin 3t)$.

However, we can restrict to just real valued functions.

The corresponding part of the *real general soln* will take the form: $e^{7x}(c_1 \cos 3x + c_2 \sin 3x)$.

Note that we have removed the i from sin, and instead multiplied both sols by arbitrary real constants. We still get two linearly independent sols, but now they're both real instead of having complex sols.

Repeated Complex Roots (of the characteristic equation):

If the roots $(5 \pm 7i)$ have multiplicity 2, the relevant part of the general soln has the form:

$$e^{5x}(c_1 \cos 7x + c_2 \sin 7x) + x e^{5x}(c_3 \cos 7x + c_4 \sin 7x).$$

Or equivalently, it's:

$$(c_1 + c_3 x) e^{5x} \cos 7x + (c_2 + c_4 x) e^{5x} \sin 7x.$$



When finding the general soln to an n th-order DEQ, you should write n arbitrary constants $(c_1, \dots, c_n)$. If you don't, you've made an error. This is a way to check your work!

Example: One soln to: $3y^{(3)} - 2y'' + 12y' - 8y = 0$, is $y = e^{\frac{2x}{3}}$. Find the general soln.

Characteristic equation: $3r^3 - 2r^2 + 12r - 8$.

Knowing that $y = e^{\frac{2x}{3}}$ is one solution, we know one factor:

$$\left(r - \frac{2}{3}\right) \text{ or } (3r - 2).$$

I have to use polynomial division! $\frac{3r^3 - 2r^2 + 12r - 8}{3r - 2}$ or $3r^3 - 2r^2 + 12r - 8 = (3r - 2) \cdot (?????)$

$$3r^3 - 2r^2 + 12r - 8 = (3r - 2)r^2 + (12r - 8)$$

Further dividing the remainder: $12r - 8 = 4(3r - 2) + 0$.

$$\text{So, we get: } 3r^3 - 2r^2 + 12r - 8 = r^2(3r - 2) + 4(3r - 2) = (3r - 2) \cdot (r^2 + 4)$$

And since $(r^2 + 4) = (r - 2i)(r + 2i)$, our two remaining sols are obtained from:

$$e^{-2ix} \text{ and } e^{2ix}, \text{ or } \dots$$

$$\cos 2x - i \sin 2x, \quad \cos 2x + i \sin 2x.$$

Linearly independent sols from above (real and imaginary parts) are: $\cos 2x$ and $\pm \sin 2x$.

Hence our general soln is: $y(x) = c_1 e^{\frac{2x}{3}} + c_2 \cos 2x + c_3 \sin 2x$.

Exercises 5.3

What did we learn?

- ♦ n th-order characteristic equation
- ♦ Differential Operators L acting on $f(x)$
- ♦ Repeated Roots
- ♦ Complex Roots



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