

Previous Lecture

- ♦ DEQs of Mechanical Systems
- ♦ Superposition of Homogeneous 2nd-order DEQ Sols
- ♦ Existence/Uniqueness for Sols to 2nd-order DEQs
- ♦ Wronskians of 2nd-order Sols
- ♦ General Sols of Homogeneous 2nd-order DEQs
- ♦ Characteristic Equation Algorithm



5.2: Gen. Sols of n th-Order Linear DEQs

Consider an n th-order linear DEQ: $y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = f(x)$. (*)

Most of the results below are merely extensions of the $n = 2$ case from the previous section, and the related proofs are nearly identical.

We can mix & match sols!

Principle of Superposition for Homogeneous DEQs Thm: Let $y_1, y_2, \dots, y_n$ be n sols of the associated *homogeneous* linear DEQ of (*) on the interval I . If $c_1, c_2, \dots, c_n$ are constants, then the linear combination $y = c_1y_1 + c_2y_2 + \dots + c_ny_n$ is also a soln on I . (not necessarily the general soln)

If you pick init-conds, a soln satisfying them exists, and it's unique!

Existence and Uniqueness for Linear DEQs Thm: Suppose that the functions $p_1, p_2, \dots, p_n$, and f are continuous on the open interval I containing the pt a . Then, given n numbers $b_0, b_1, \dots, b_{n-1}$, the DEQ (*) has a unique (that is, one and only one) soln on the *entire* interval I that satisfies the n init-conds: $y(a) = b_0, y'(a) = b_1, \dots, y^{(n-1)}(a) = b_{n-1}$.

Independence

How do we determine whether n sols to our DEQ are linearly independent, so that we might form a general soln?

Recall that with two functions, we needed $f_1 = cf_2$ on I for *dependence*, or $f_1 \neq cf_2$ for *independence*.

Also, recall that w/ n vectors $\vec{v}_i$, *dependence* was assured if $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$ with $c_1, c_2, \dots, c_n$, not all zero.

And *independence* was assured if $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$ required that $c_1, c_2, \dots, c_n$, all be zero.

But now recall that real-valued functions are "vectors" in a vectors space. Therefore, we have the following.

Definition — Linear Dependence of Functions: The n functions $f_1, f_2, \dots, f_n$ are linearly dependent on the interval I if there exists constants $c_1, c_2, \dots, c_n$, not all zero, such that $c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$ on I ; that is, $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for all x in I .

Therefore, just as w/ n -tuple vectors, if functions are dependent, we can solve for one of the functions in terms of a linear combination of the others.

Wronskian of Homogeneous Solutions Thm: Suppose that $y_1, y_2, \dots, y_n$ are n sols of the associated *homogeneous* linear DEQ of $(*)$ on an open interval I , where each p_i is continuous. Let $W(x) = W(y_1, y_2, \dots, y_n)$.

- ♦ If $y_1, y_2, \dots, y_n$ are dependent, then $W(x) \equiv 0$, at each pt x in I .
- ♦ If $y_1, y_2, \dots, y_n$ are independent, then $W(x) \neq 0$, at each pt x in I .

Thus, there are just two possibilities: either $W \equiv 0$ everywhere on I , or $W \neq 0$ everywhere on I (so you need only check one pt).

In the above thm, let’s prove the first bullet pt: if $y_1, y_2, \dots, y_n$ are linearly dependent, then $W(x) \equiv 0$, at each pt x in I .

Proof: Since we can assume dependence, we have that $c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$ holds at each pt x in I for some choice of $c_1, c_2, \dots, c_n$, not all zero.

Next, differentiate this equation $n - 1$ times in succession, obtaining:

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) &= 0 \\ c_1 y_1'(x) + c_2 y_2'(x) + \dots + c_n y_n'(x) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \dots + c_n y_n^{(n-1)}(x) &= 0 \end{aligned}$$

which still holds at each pt x in I .

Observe that the unknowns in the above system are the c_i . Therefore, this can be rewritten as:

$$\mathbf{A} \vec{c} = \vec{0}, \text{ where } \vec{c} := (c_1, \dots, c_n) \text{ and } \mathbf{A} := \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}.$$

Now recall that a homogeneous $n \times n$ linear system of equations has a nontrivial soln *iff* its coefficient matrix $\mathbf{A}$ is not invertible, and since dependence gave us a nontrivial $\vec{c}$, $\mathbf{A}$ is not invertible.

We also learned non-invertibility only happens when the determinant of the coefficient matrix $|\mathbf{A}|$ is zero.

In this case, the determinant is recognizable as the Wronskian $W(x)$ of the y_i .

Thus, $W(x) \equiv 0$, as we wished to prove. ■

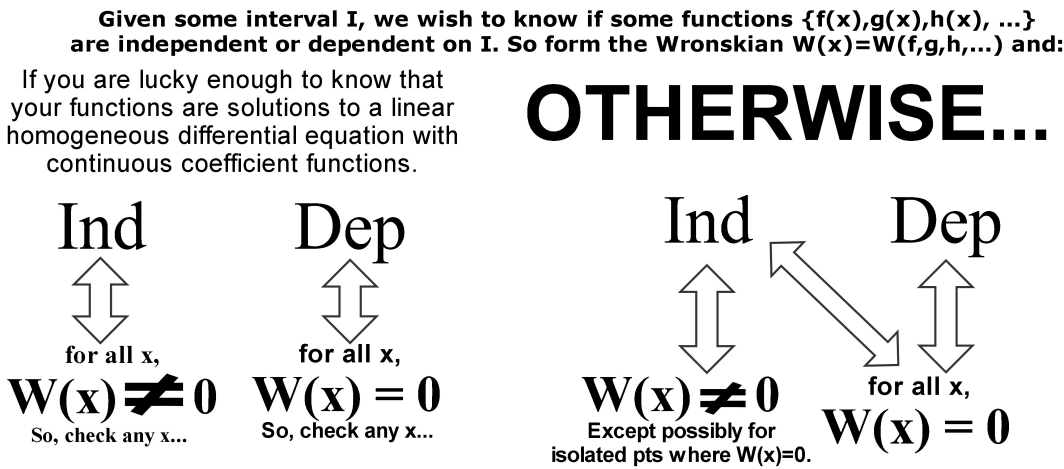
The above works when the functions you're examining are sols to a homogeneous DEQ. But what if you wish to know the independence of some functions (on some open interval I) which are *not* known to be sols to a DEQ?

- ◆ If the Wronskian is nonzero (except possibly at some isolated pts, so you can't just check one pt), then the functions are independent.
- ◆ Also, if the functions are dependent, then you know that the Wronskian is always zero.

However, the opposite implications do not follow. In other words,

- ◆ If you only know that the functions are independent, you know nothing about the Wronskian (could be zero or nonzero).
- ◆ And if you only know that the Wronskian is zero everywhere, you know nothing about the independence of the functions.

Here is a graphic that might clarify (or confuse) things for you...



Example: Show that the functions $y_1(x) = x$, $y_2(x) = 56x \ln x$, $y_3(x) = x(\ln x)^2$ are linearly independent on $(0, \infty)$.

These aren't sols to a DEQ, so we can't just check one pt!

$$\begin{aligned} W(x) &= \begin{vmatrix} x & x \ln x & x(\ln x)^2 \\ 1 & \ln x + 1 & (\ln x)^2 + 2 \ln x \\ 0 & \frac{1}{x} & \frac{2 \ln x + 2}{x} \end{vmatrix} = x \left((\ln x + 1) \frac{2 \ln x + 2}{x} - \frac{(\ln x)^2 + 2 \ln x}{x} \right) - \left(x \ln x \cdot \frac{2 \ln x + 2}{x} - x(\ln x)^2 \cdot \frac{1}{x} \right) \\ &= ((\ln x + 1)(2 \ln x + 2) - (\ln x)^2 - 2 \ln x) - (\ln x \cdot (2 \ln x + 2) - (\ln x)^2) \\ &= (2 \ln x(\ln x + 1) + 2(\ln x + 1) - (\ln x)^2 - 2 \ln x) - \ln x(\ln x + 2) \\ &= (\ln x)^2 + 2 \ln x + 2 - (\ln x)^2 - 2 \ln x = 2 \neq 0 \text{ (for any } x\text{)}. \end{aligned}$$

Therefore, the functions are linearly independent.

Continuing w/our **n th-order** linear DEQ: $y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = f(x)$. (*)

So now that we know our sols are independent, we can form a general solution...

General Sols of Homogeneous n th-order DEQs Thm: Let's say you know that $y_1, y_2, \dots, y_n$ are linearly independent sols of (*)'s associated *homogeneous* DEQ on an open interval I , where the p_i are continuous. If Y is any soln whatsoever to the homogeneous DEQ, then there exist numbers $c_1, c_2, \dots, c_n$ such that $Y(x) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ for all x in I . (i.e., all other sols can be characterized as a linear combination of these linearly independent ones)

I didn't previous share the 2D version of the following thm. It tells us how to form sols to *non-homogeneous* DEQs.

Sols to Non-Homogeneous n th-order DEQs Thm: Let's say you know that y_p is a **particular soln** for the *non-homogeneous* DEQ $(*)$ on an open interval I , where the p_i and f are continuous. And suppose $y_1, y_2, \dots, y_n$ are linearly independent sols of $(*)$'s associated *homogeneous* DEQ. Then if $Y(x)$ is any soln whatsoever to the non-homogeneous DEQ, then there exist numbers $c_1, c_2, \dots, c_n$ such that for all x in I we have: $Y(x) = y_p + (c_1 y_1 + c_2 y_2 + \dots + c_n y_n)$.

Proof: Let Y and y_p be sols to $(*)$.

Define $y_c := Y - y_p$. Substituting this into the $(*)$'s associated *homogeneous* DEQ:

$$\begin{aligned} & (Y - y_p)^{(n)} + p_1(x)(Y - y_p)^{(n-1)} + \dots + p_{n-1}(x)(Y - y_p)' + p_n(x)(Y - y_p) \\ &= \left(Y^{(n)} + p_1(x)Y^{(n-1)} + \dots + p_{n-1}(x)Y' + p_n(x)Y \right) - \left(y_p^{(n)} + p_1(x)y_p^{(n-1)} + \dots + p_{n-1}(x)y_p' + p_n(x)y_p \right) \\ &= f(x) - f(x) = 0. \end{aligned}$$

Therefore, $y_c = Y - y_p$ is a soln to $(*)$'s associated *homogeneous* DEQ.

Recall that *any* homogeneous soln can be written as a linear combo: $y_c = c_1 y_1 + \dots + c_n y_n$.

But if we rearrange $y_c = Y - y_p$, we find $Y = y_p + y_c = y_p + (c_1 y_1 + \dots + c_n y_n)$.

Recall our choice of Y as a soln to the non-homogeneous DEQ was arbitrary.

So we have shown that a *general soln* Y of the non-homogeneous DEQ is the sum of its complementary function y_c and *any* particular soln y_p . ■

Notice the trivial soln $y(x) \equiv 0$, is a soln to the homogenous: $y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0$.

Furthermore, $y(x) \equiv 0$ is the unique (only) soln to the DEQ that satisfies the trivial init-conds:
 $y(a) = 0, \quad y'(a) = 0, \quad \dots, \quad y^{(n-1)}(a) = 0$.

Exercises 5.2

What did we learn?

- ♦ n th-order (general) versions of 1st-order thms

