

Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ♦ What are Vectors
- ♦ What's a Vector Space
- ♦ When are Vectors Dependent/Independent
- ♦ Bases
- ♦ Subspaces



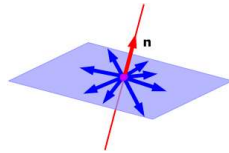
4.2: The Vector Space $\mathbb{R}^n$ and Subspaces

Dimensions $n > 3$, what are they good for?

Space-Time (x, y, z, t) , multiple particle systems, robotics, stock portfolios, local properties (pressure, temperature, color, velocity, etc.).



Subspaces

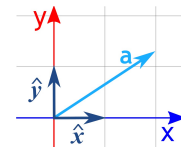


Recall Subspaces: Given vector space V (e.g., $\mathbb{R}^3$ in image above), then W (the plane in the image) is a subset of V , and is called a subspace if:

- ♦ $W \neq \emptyset$ (it contains at least one vector), (W is **nonempty**)
- ♦ Given $\vec{u}, \vec{v} \in W$, we have $\vec{u} + \vec{v} \in W$, (W is closed under **addition**)
- ♦ Given $c \in \mathbb{R}$, we have $c\vec{u} \in W$. (W is closed under **scalar multiplication**)

And therefore $\vec{0} \in W$. Why? (this is sometimes a quick way to discover that a subset is *not* a subspace)

Also, convince yourself that the x -axis and the y -axis (just the axes themselves, no other points), joined together as a subset of $\mathbb{R}^3$, does not constitute a subspace.



❗ Another reason subspaces are important is that the solutions to homogeneous linear systems are subspaces!

Homogeneous Soln Subspace Thm: For $A^{m \times n}$, the soln set of $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

Proof: Let W denote the solution set of the system. If $\vec{u}$ and $\vec{v}$ are vectors in W , then $A\vec{u} = A\vec{v} = \vec{0}$.

$$\text{Hence: } A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}.$$

Thus the sum $\vec{u} + \vec{v}$ is also in W , and hence W is closed under addition.

Next, if $c \in \mathbb{R}$, then $\mathbf{A}(c\vec{u}) = c(\mathbf{A}\vec{u}) = c\vec{0} = \vec{0}$.

Thus $c\vec{u}$ is in W . Hence W is also closed under scalar multiplication.

$\vec{0}$ is the trivial soln, so $W \neq \emptyset$. Therefore, W is a subspace of $\mathbb{R}^n$. ■

But this is not true for nonhomogeneous systems.

Nonhomogeneous System Soln Thm: The soln set of nonhomogeneous $\mathbf{A}\vec{x} = \vec{b}$ is *never* a subspace.

Proof: Let's do proof by contradiction. Let $\vec{u}$ be a solution in W , with W being the set of solutions to the nonhomogeneous system above.

And let us make the dubious assumption that W is a subspace.

Let $c = 0 \in \mathbb{R}$. By closure under scalar multiplication, $c\vec{u} = 0\vec{u} = \vec{0}$ is a solution.

Therefore, $\mathbf{A}\vec{0} = \vec{0} = \vec{b}$. But we assumed the system was nonhomogeneous: $\vec{b} \neq \vec{0}$.

So we have a contradiction, and our assumption that the set of solutions W was a subspace must have been incorrect. ■

An alternative proof is provided in the book.

Example (problem 1 of exercises): For the following system of equations, find two solution vectors $\vec{u}$ and $\vec{v}$ (a basis) such that the **solution space** is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$.

$$\begin{aligned}x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\2x_1 - x_2 + x_3 + 7x_4 &= 0 \\x_1 + 2x_2 + 3x_3 + 11x_4 &= 0\end{aligned}$$

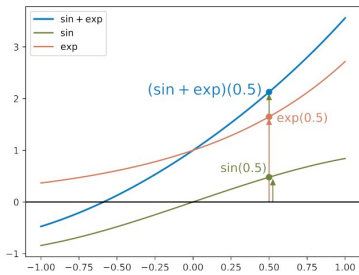
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \xrightarrow{\text{trust me}} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_2 = -s - 3t$, and $x_1 = -s - 5t$. So ...

Solution is: $\vec{x} = (x_1, x_2, x_3, x_4) = (-s - 5t, -s - 3t, s, t)$

$$= (-s, -s, s, 0) + (-5t, -3t, 0, t) = s\vec{u} + t\vec{v}, \text{ where } \vec{u} = (-1, -1, 1, 0) \text{ and } \vec{v} = (-5, -3, 0, 1).$$

Function Spaces



Vectors are not the only things that form vector spaces!
Indeed, some folks don't even refer to these spaces as vector spaces, but rather as *linear spaces*.
For now, we will learn about another type of vector/linear space, namely function spaces.

Polynomial spaces satisfy vector space properties:

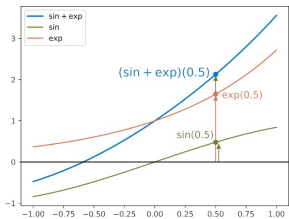
Let $P_3(x)$ be the set of 3rd degree polynomials: $\{a_0 + a_1x + a_2x^2 + a_3x^3\}$. Given p_1, p_2, p_3 in $P_3(x)$, observe that:

- ◆ $p_1 + p_2 = p_2 + p_1$ [additive commutivity]
- ◆ $p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$ [additive associativity]
- ◆ $p_1 + 0 = 0 + p_1 = p_1$ [additive identity, where 0 is the trivial polynomial]
- ◆ $p_1 + (-p_1) = (-p_1) + p_1 = 0$ [additive inverse]
- ◆ $(c + k)p_1 = cp_1 + kp_1$, for any real c, k . [distributivity]
And $k(p_1 + p_2) = kp_1 + kp_2$, for any real k
- ◆ $c(kp_1) = (ck)p_1$ [scalar multiplicative associativity]
- ◆ $1(p_1) = p_1$ [scalar multiplicative identity]

Function Spaces (Vector Spaces of Functions)

Let $F = \{ \text{all real valued functions on } \mathbb{R} \}$; includes all polynomials, trig functions, exponentials, etc... ...

Observe for $\mathbf{f}, \mathbf{g}$ in F , we have: $(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$ and $(c\mathbf{f})(x) = c(\mathbf{f}(x))$.



The properties of a vector space follow since these functions are real valued.

What is the zero "vector"? $z(x)$...

Vector Space of $m \times n$ Matrices

Example: 2×2 matrices. Matrices like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with zero element $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and multiplicative identity element $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercises 4.2

What did we learn?

- ◆ Higher dimensional spaces/subspaces
- ◆ Subspaces as solutions to homogeneous systems
- ◆ Function spaces



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Materials for Other Courses Found at **MathTalker.org**