

Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ◆ 1st-Order DEQs Substitution Methods
- ◆ Scalar Homogeneity: $y' = F(\frac{y}{x})$
- ◆ Bernoulli DEQs: $y' + yP(x) = y^n Q(x)$
- ◆ Exact DEQs: $M(x,y) + N(x,y)y' = 0$

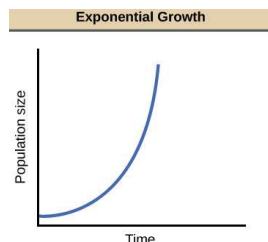


2.1: Mathematical Models and Numerical Methods

Population Models

Our previous (exponential) population model was: $\frac{dP}{dt} = kP$, with $P(t) = P_0 e^{kt}$.

where the **birth and death rates** (β, δ) were constant,
and $k := \beta - \delta$.



However, the following model is more realistic, with *variable* death and birth rates.

(but still assumes no immigration or emigration)

The **General Population Equation**:

$$\frac{dP(t)}{dt} = \beta(t)P(t) - \delta(t)P(t) = (\beta(t) - \delta(t))P(t),$$

where $\beta(t)$ and $\delta(t)$ are the (variable) rate of births and deaths (respectively) per person (or animal), occurring at time t .

Notice that when β and δ are constant in this equation, we retrieve: $\frac{dP}{dt} = kP$, where $k := \beta - \delta$.

Bounded Populations

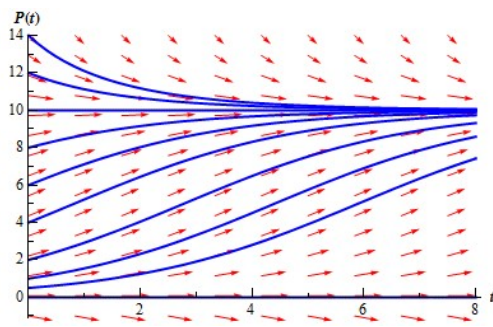
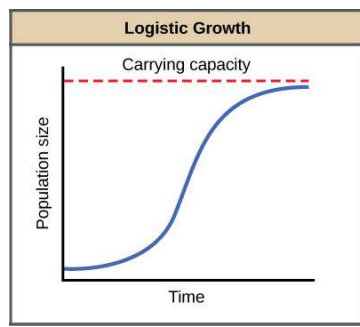
A common population model, called the **logistic equation**, occurs when β is *linearly decreasing* wrt P

(which means: $\beta = c - kP$, with $c, k > 0$), and the death rate δ is constant. Substituting this in, we find:

$$\frac{dP}{dt} = (\beta - \delta)P = (c - kP - \delta)P = kP\left(\frac{c-\delta}{k} - P\right) = kP(M - P), \text{ where } M = \frac{c-\delta}{k}.$$

The reason we notate it this way is to isolate an important quantity M , which we can see on a graph below as the "**limiting population**" or "**carrying capacity**."

Logistic Equation: $\frac{dP}{dt} = kP(M - P)$. (models: finite resources, interspecies competition, disease, etc.)



Using your newly acquired skill of separation-of-variables, you can now show that the init-val problem:

$$\frac{dP}{dt} = kP(M - P), \quad P(0) = P_0 \text{ has the solution } P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}. \text{ (try this as an exercise!)}$$

What happens as $t \rightarrow \infty$ if the initial population P_0 has:

$$P_0 = M?$$

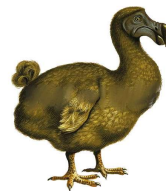
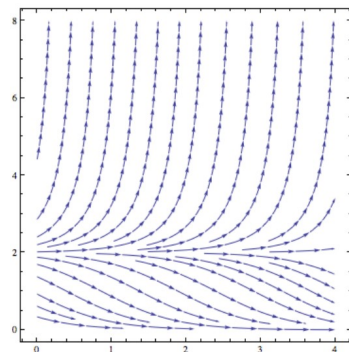
$$P_0 > M?$$

$$P_0 < M?$$

Observe $\lim_{t \rightarrow \infty} P(t) = \frac{MP_0}{P_0 + 0} = M$.

Explosion or Extinction Model: $\frac{dP}{dt} = kP(P - M)$.

(it models species near extinction)



Dodo

Observe that if you are below a certain population level, you are doomed to extinction!

Partial Fractions to Solve DEQs

Example: Separate variables to solve the init-val problem: $\frac{dP}{dt} = 7P(P - 13)$, $P(0) = 17$.

$$\frac{1}{P(P-13)} dP = 7dt, \text{ if } P \neq 13 \text{ and } P \neq 0.$$

Can $P(t) \equiv 13$ or $P(t) \equiv 0$, be sols with our init-cond $P(0) = 17$?

$$\int \frac{1}{P(P-13)} dP = \int 7dt$$

We would prefer to integrate something more like $\int \frac{A}{P} + \frac{B}{P-13} dP = \int 7dt$.

Partial Fractions:

Observe that $\frac{1}{P(P-13)} = \frac{A}{P} + \frac{B}{P-13}$, when $1 = A(P-13) + BP$. (multiplied through by denom $P(P-13)$)

Collecting powers of P : $(A+B)P - 13A = 1$,

Comparing powers of P : $A+B = 0$ and $-13A = 1$,

$$\text{so } A = -\frac{1}{13}, \text{ and } B = \frac{1}{13}.$$

$$\text{Therefore: } \int \frac{1}{P(P-13)} dP = \int \frac{-\frac{1}{13}}{P} + \frac{\frac{1}{13}}{P-13} dP.$$

$$\text{And our equation becomes: } -\frac{1}{13} \int \frac{1}{P} - \frac{1}{P-13} dP = \int 7dt$$

$$\Rightarrow \int \left(\frac{1}{P} - \frac{1}{P-13} \right) dP = -91 \int dt$$

$$\Rightarrow \ln|P| - \ln|P-13| = -91t + c$$

$$\Rightarrow \ln \left| \frac{P}{P-13} \right| = -91t + c$$

$$\Rightarrow \frac{P}{P-13} = Ce^{-91t}, \text{ where } C \neq 0. \quad \text{Then ...?}$$

$$\text{Since } P(0) = 17, \quad \frac{17}{17-13} = Ce^{-91 \cdot 0} = C, \quad \Rightarrow C = \frac{17}{4}.$$

$$\text{So, } \frac{P}{P-13} = \frac{17}{4} e^{-91t}.$$

Solve explicitly for P ?

$$\Rightarrow 4P = 17(P-13)e^{-91t}$$

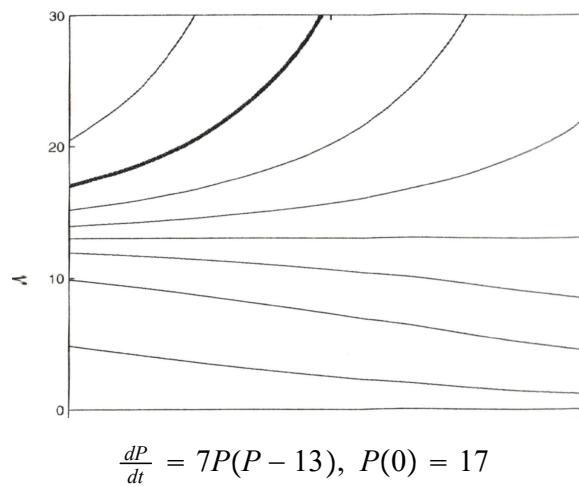
$$\Rightarrow 4P = 17Pe^{-91t} - 221e^{-91t}, \quad \Rightarrow \quad 4P - 17Pe^{-91t} = -\frac{221}{e^{91t}}$$

$$\Rightarrow P(4 - 17e^{-91t}) = -\frac{221}{e^{91t}}, \quad \Rightarrow \quad P = -\frac{221}{(4 - 17e^{-91t})e^{91t}},$$

(when $4 - 17e^{-91t} \neq 0$, which occurs when $t \neq -\frac{1}{91} \ln \frac{4}{17} \approx 0.016$)

$$\Rightarrow P(t) = \frac{221}{17 - 4e^{91t}}.$$

Here are the graphs of several sols of the given DEQ. Highlighted is the indicated particular soln.



Exercises 2.1

What did we learn?

- ♦ Population models with variable birth/death rates
- ♦ Logistic Equation: $\frac{dP}{dt} = kP(M - P)$
- ♦ Partial Fractions: $\frac{1}{f(x)g(x)} = \frac{A}{f(x)} + \frac{B}{g(x)}$



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Materials for Other Courses Found at **MathTalker.org**